

Naturality of FHT isomorphism

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Abstract

Freed, Hopkins and Teleman constructed an isomorphism (we call it FHT isomorphism) between twisted equivariant K -theory of compact Lie group G and the “Verlinde ring” of the loop group of G ([FHT1] [FHT2] [FHT3]). However, this isomorphism does not hold naturality with respect to group homomorphisms. We construct two “quasi functors” $t.e.K$ and RL so that FHT isomorphism is natural transformation between two “quasi functors” for tori, that is, we construct two “induced homomorphisms” $f^\# : t.e.K(T', \tau) \rightarrow t.e.K(T, f^*\tau)$ and $f^! : RL(T', \tau) \rightarrow RL(T, f^*\tau)$ for $f : T \rightarrow T'$ whose tangent map is injective. In fact, we construct another quasi functor $char$ and verify that three quasi functors are naturally isomorphic.

Moreover, we extend the quasi functor $t.e.K$ and $char$ to compact connected Lie group with torsion-free π_1 and homomorphism $f : H \rightarrow G$ satisfying the decomposable condition, and verify that they are isomorphic. This is a generalization of naturality of $K_G^{\tau+rank(G)}(G) \cong char(G, \tau)$ verified in [FHT1].

Introduction

Study of loop groups of compact Lie groups is extensively developed especially from the view point of representation theory. On the other hand, twisted K -theory was introduced by Donovan and Karoubi [Karoubi], and interest in it was rekindled by its appearance in string theory. Freed, Hopkins and Teleman connected them ([FHT2]). In this paper, we study functorial aspects of these two objects.

Let us start from the representation theory of loop groups. Positive energy representations are special cases, defined by S^1 symmetry of loop groups. S^1 acts on loop groups via transformation of the parameter. When we make S^1 act on loop groups, we write S^1 as T_{rot} . Let G be a compact connected Lie group with torsion-free π_1 and \mathcal{H} be a separable complex Hilbert space as a representation space.

Definition 0.1 ([PS]). *A positive energy representation is a continuous projective representation $\rho : LG \rightarrow PU(\mathcal{H})$ satisfying the following conditions.*

(1) The action of \mathbb{T}_{rot} on LG lifts to the associated central extension by $U(1)$

$$LG^\tau := LG \times_{PU(\mathcal{H})} U(\mathcal{H}).$$

Therefore, we can define the new Lie group $LG^\tau \rtimes \mathbb{T}_{rot}$.

(2) ρ lifts to $\hat{\rho} : LG^\tau \rtimes \mathbb{T}_{rot} \rightarrow U(\mathcal{H})$.

(3) There are no vectors v on which rotation $e^{i\theta} \in \mathbb{T}_{rot}$ acts by multiplication by $e^{in\theta}$ with sufficient small n .

Remark 0.2. A central extension τ is also called level. A representation is at level τ if the induced central extension $LG \times_{PU(\mathcal{H})} U(\mathcal{H})$ is isomorphic to LG^τ .

The direct sum of two representations at level τ is also at level τ . So finitely reducible positive energy representations at level τ form a semigroup under the direct sum.

Definition 0.3. The representation group $R^\tau(LG)$ of LG at level τ is the Grothendieck completion of this semigroup.

Our second main object is twisted equivariant K -theory.

Let G be a compact Lie group, X be a topological space. We suppose that G acts on X continuously. Let τ be a G -equivariant twisting over X and $\{K_G^{\tau+k}(X)\}_{k \in \mathbb{Z}}$ be the τ -twisted K -groups, which are defined in Section 1.

Let G have torsion-free π_1 and τ be a “positive” central extension of LG , which is defined in Section 1. A central extension τ defines a G -equivariant twisting over G ([AS]), where G acts on itself by conjugation. Freed, Hopkins and Teleman constructed an isomorphism between $R^\tau(LG)$ and $K_G^{\tau+\sigma+rank(G)}(G)$, where σ is the special central extension of LG called the spin extension.

Theorem 0.4 ([FHT2]). *In this situation, we have an isomorphism*

$$FHT_G : R^\tau(LG) \rightarrow K_G^{\tau+\sigma+rank(G)}(G).$$

We call it FHT isomorphism.

However, this isomorphism does not hold naturality. In this paper, we study naturality of FHT isomorphisms for tori.

Let us consider pull backs for two cases. Let $f : T' \rightarrow T$ be a smooth group homomorphism between two tori T and T' . We suppose that the tangent map df is injective, we call such a map a local injection.

For the case of representation group, the pull back of finitely reducible representations along f is not finitely reducible when $\dim(T) \neq \dim(T')$ (Corollary 5.6), that is, $f^* : R^\tau(LT) \rightarrow R^{f^*\tau}(LT')$ cannot be defined by use of the pull back of representations. For the case of twisted equivariant K -theory, the K -theoretical pull back must be zero if $\dim(T) - \dim(T')$ is odd (Corollary 2.10).

Motivated by these observations, we modify pull backs so that the map FHT_T holds naturality. That is, we construct “induced homomorphisms”

$$f^\# : K_T^{\tau+\dim(T)}(T) \rightarrow K_{T'}^{f^*\tau+\dim(T')}(T')$$

$$f^! : R^\tau(LT) \rightarrow R^{f^*\tau}(LT').$$

By use of these induced homomorphisms, we verify the following main theorem.

Theorem 0.5. *The following commutative diagram holds.*

$$\begin{array}{ccc} K_T^{\tau+\dim(T)}(T) & \xrightarrow{f^\#} & K_{T'}^{f^*\tau+\dim(T')}(T') \\ FHT_T \downarrow & & FHT_{T'} \downarrow \\ R^\tau(LT) & \xrightarrow{f^!} & R^{f^*\tau}(LT') \end{array}$$

This theorem is verified by use of another objects $char(T, \tau)$, $char(T', f^*\tau)$ and homomorphism $char(f)$, which are defined later.

In fact, we verify the following two theorems. Vertical arrows are defined later.

Theorem 0.6. *The following commutative diagram holds.*

$$\begin{array}{ccc} K_T^{\tau+\dim(T)}(T) & \xrightarrow{f^\#} & K_{T'}^{f^*\tau+\dim(T')}(T') \\ M.d.T \downarrow & & M.d.T' \downarrow \\ char(T, \tau) & \xrightarrow{char(f)} & char(T', f^*\tau) \end{array}$$

Theorem 0.7. *The following commutative diagram holds.*

$$\begin{array}{ccc} R^\tau(LT) & \xrightarrow{f^!} & R^{f^*\tau}(LT') \\ l.w.T \downarrow & & l.w.T' \downarrow \\ char(T, \tau) & \xrightarrow{char(f)} & char(T', f^*\tau) \end{array}$$

We can extend Theorem 0.6 to $f : H \rightarrow G$ satisfying the decomposable condition (Section 2), where H and G can be not tori.

Theorem 0.8. *The following commutative diagram holds.*

$$\begin{array}{ccc} K_G^{\tau+\text{rank}(G)}(G) & \xrightarrow{f^\#} & K_H^{f^*\tau+\text{rank}(H)}(H) \\ M.d.G \downarrow & & M.d.H \downarrow \\ char(G, \tau) & \xrightarrow{char(f)} & char(H, f^*\tau) \end{array}$$

Let us explain the details and the backgrounds.

Firstly, we explain the details on representation groups. Let us start from the representation theory of compact Lie groups (Cartan-Weyl theory). Before that, we deal with tori. Representation theory for tori is fundamental. Let T be an n dimensional torus. Let us recall an explicit description of representation ring of T

$$R(T) \cong \mathbb{Z}[X_1, X_2, \dots, X_n, (X_1 X_2 \cdots X_n)^{-1}],$$

where $R(T)$ is the Grothendieck completion of the semiring which finite dimensional representations of T form. X_j corresponds to the projection onto the j 'th factor $p_j : T \cong (U(1))^n \rightarrow U(1)$. This isomorphism is verified as follows. By Schur's lemma, any irreducible unitary representations are 1 dimensional. So we can identify the set of isomorphism classes of irreducible representations with the set of homomorphisms from T to $U(1)$. We write it as $\text{Hom}(T, U(1))$. We can rewrite it by taking tangent maps. Let $\Pi_T := \ker(\exp : \mathfrak{t} \rightarrow T)$ and $\Lambda_T := \text{Hom}(\Pi_T, \mathbb{Z})$. We have a canonical isomorphism $\text{Hom}(T, U(1)) \cong \Lambda_T$ by taking tangent maps. Since Λ_T is isomorphic to the \mathbb{Z} free module generated by X_1, X_2, \dots, X_n , the isomorphism $R(T) \cong \mathbb{Z}[X_1, X_2, \dots, X_n, (X_1 X_2 \cdots X_n)^{-1}]$ holds. If we regard $R(T)$ just as a group, it is isomorphic to the \mathbb{Z} free module generated by Λ_T . So we write it as $\mathbb{Z}[\Lambda_T]$ also.

Let G be a compact, connected, simply connected and simple Lie group, and T be a chosen maximal torus. Let us recall a very beautiful correspondence

$$\{\text{isomorphism classes of irreducible representations}\} \cong \{\text{dominant weights}\}.$$

Since for any weight λ , there exists $w \in W(G)$ such that $w.\lambda$ is a dominant weight, we have an isomorphism $R(G) \cong R(T)^{W(G)}$, the invariant subring of $R(T)$. Moreover, we have an isomorphism as groups

$$R(G) \cong \mathbb{Z}[\Lambda_T / W(G)],$$

where the right hand side $\mathbb{Z}[\Lambda_T / W(G)]$ is the \mathbb{Z} free module generated by the set $\Lambda_T / W(G)$. Let us explain this theorem. An isomorphism class of irreducible representation of G is completely determined by the restriction to T . We can define a lexicographic order by taking an isomorphism $T \cong (U(1))^n$. Let $\rho : G \rightarrow U(V)$ be an irreducible representation. We can decompose V as irreducible representation of T . This decomposition determines the $W(G)$ -invariant weighted finite subset in Λ_T , where $W(G)$ is the Weyl group of G . This finite subset is determined by the maximum element (Cartan-Weyl's highest weight theory), which is "dominant". Moreover, if we are given a dominant weight, we can define an irreducible representation whose highest weight is the given one (Borel-Weil theory).

An analogue of Cartan-Weyl theory for loop groups is studied in [PS]. Let us state a difficulty when we try to follow the above construction of compact Lie group case. Let T be a torus and LT be the set of smooth maps from S^1 to T , which is the loop group of T . The group structure of LT is defined by the multiplication of T pointwisely. However, the above mechanism for compact Lie group does not hold in this case, since the size of maximal connected commutative group LT_0 is too large, where LT_0 is the identity component of LT . Precisely, it is too difficult to deal with all representations. In fact, it has not been known to classify all representations of LT . In this paper, we focus on positive energy representations.

Positive energy representations at level τ has been classified ([PS]). In order to describe it for tori, we need to introduce some terminologies.

Let τ be a positive central extension of LT . We can restrict it to the set of constant loops, which is naturally identified with T .

$$\begin{array}{ccccccc}
1 & \longrightarrow & U(1) & \xrightarrow{i} & LT^\tau & \xrightarrow{p} & LT \longrightarrow 1 \\
\parallel & & \parallel & & \uparrow & & \uparrow \\
1 & \longrightarrow & U(1) & \xrightarrow{i} & T^\tau & \xrightarrow{p} & T \longrightarrow 1
\end{array}$$

$\Lambda_T^\tau := \{\rho : T^\tau \rightarrow U(1) \mid \rho \circ i = id\}$ is the set of τ -twisted irreducible representations of T^τ . A central extension τ defines an action of Π_T on Λ_T^τ (Section 1).

Theorem 0.9 ([PS]). *Isomorphism classes of positive energy representations at level τ are parametrized by the set of Π_T -orbits in Λ_T^τ . That is, we have an isomorphism*

$$l.w._T : R^\tau(LT) \xrightarrow{\cong} \mathbb{Z}[\Lambda_T^\tau / \Pi_T],$$

where $\mathbb{Z}[\Lambda_T^\tau / \Pi_T]$ is the \mathbb{Z} free module generated by the set Λ_T^τ / Π_T .

The notation *l.w.* comes from the “lowest weight”. It is known that we can take a maximal torus of $LT^\tau \rtimes \mathbb{T}_{rot}$ as $T^\tau \times \mathbb{T}_{rot}$ if τ is positive. By formal computation in $LT^\tau \rtimes \mathbb{T}_{rot}$, Weyl group of $LT^\tau \times \mathbb{T}_{rot}$ is isomorphic to Π_T . Therefore, we can regard the above theorem as an analogue of the isomorphism $R(G) \cong R(T)^{W(G)} \cong \mathbb{Z}[\Lambda_T / W(G)]$.

Secondly, we explain the details on twisted equivariant K -theory.

Twisted K -theory was introduced by Donovan and Karoubi in order to extend Thom isomorphism for non K -orientable vector bundle [Karoubi]. Later, it was extended by Rosenberg [AS]. It has an equivariant version just like untwisted cases.

Let us start from untwisted equivariant K -theory. Equivariant K -theory is concerned with representation theory ([Segal]). Throughout this paper, we deal with compactly supported K -theory. Let X be a topological space and G be a Lie group acting on X . If X is a compact Hausdorff space, isomorphism classes of G -equivariant vector bundles form a semigroup under the direct sum, and equivariant K -group $K_G(X)$ is defined by the Grothendieck completion of this semigroup. Generally, $K_G(X)$ is the set of G -equivariant homotopy classes of G -equivariant family of Fredholm operators parametrized by X . When an action of G is special, we can describe $K_G(X)$ easily. When it is trivial, $K_G(X) \cong K(X) \otimes R(G)$, and when it is free, $K_G(X) \cong K(X/G)$.

Our second main object is twisted version of it. We start from non-equivariant twisted K -theory. A twisting over X is an element of the group $H^3(X)$. It classifies $PU(\mathcal{H})$ principal bundles over X , where \mathcal{H} is an infinite dimensional separable complex Hilbert space, since the functor H^3 has the classifying space $K(\mathbb{Z}, 3)$, and the homotopy equivalences $PU(\mathcal{H}) \cong K(\mathbb{Z}, 2)$ and $BPU(\mathcal{H}) \cong K(\mathbb{Z}, 3)$ hold, where $K(\mathbb{Z}, 3)$ and $K(\mathbb{Z}, 2)$ are Eilenberg MacLane spaces. Therefore, if $\tau \in H^3(X)$ is given, we have an associated $PU(\mathcal{H})$ principal bundle Q . And it determines a $\mathbb{P}(\mathcal{H})$ bundle $P := Q \times_{PU(\mathcal{H})} \mathbb{P}(\mathcal{H})$. We call such P a projective bundle, that is, it is a $\mathbb{P}(\mathcal{H})$ bundle whose structure group is $PU(\mathcal{H})$.

Then we have a classifying bundle $Fred(P)$ of twisted K -group $K^{\tau+0}(X)$, where $Fred(P)|_x$ is the set of Fredholm operators on \mathcal{H}_x , a chosen lift of the fiber $P|_x$. Twisted K -theory is defined as follows.

Definition 0.10. *The space of sections is defined by*

$$\Gamma_c(X, Fred(P)) := \{s : X \rightarrow Fred(P) | \text{supp}(s) \text{ is compact}\},$$

where $\text{supp}(s) := \{x \in X | s_x \text{ is not invertible}\}$ is the support of s .

Twisted K -theory is defined by

$$K^{\tau+0}(X) := \pi_0(\Gamma_c(X, Fred(P)))$$

$$K^{\tau+k}(X) = K^{\tau-k}(X) := K^{p^*\tau+0}(X \times \mathbb{R}^k),$$

where $p : X \times \mathbb{R}^k \rightarrow X$ is the projection onto the first factor.

Remark 0.11. *Twisted K -theory is a two periodic generalized cohomology theory ([AS]).*

Equivariant version of twisted K -theory is defined just like untwisted cases. Let G be a compact Lie group acting on a topological space X continuously. If we replace “projective bundle” with “ G -equivariant projective bundle”, “sections” with “ G -equivariant sections” and “homotopy” with “ G -equivariant homotopy”, we obtain the twisted equivariant K -theory $K_G^{\tau+k}(X)$, where $\tau \in H_G^3(X)$ is a G -equivariant twisting. Since this is a generalized cohomology theory for G -spaces, we have some tools to compute. In fact, Freed, Hopkins and Teleman computed $K_G^{\tau+k}(G)$ using spectral sequences or a “Mackey decomposition” ([FHT1], [FHT3]) for τ which is induced by a positive central extension of LG , where G acts on itself by conjugation. For simplicity, we deal with tori here. While this action of T on itself is trivial, an action on a projective bundle can be non-trivial.

Theorem 0.12 ([FHT1], [FHT3]). *The following isomorphism holds.*

$$K_T^{\tau+k}(T) \cong \begin{cases} 0 & (k \equiv \dim(T) + 1 \pmod{2}) \\ \mathbb{Z}[\Lambda_T^\tau / \Pi_T] & (k \equiv \dim(T) \pmod{2}). \end{cases}$$

The action of Π_T on Λ_T^τ coincides with the one defined by the central extension τ . $\mathbb{Z}[\Lambda_T^\tau / \Pi_T]$ is the \mathbb{Z} free module generated by the set Λ_T^τ / Π_T .

We write the isomorphism of the above theorem as

$$M.d.T : K_T^{\tau+\dim(T)}(T) \rightarrow \mathbb{Z}[\Lambda_T^\tau / \Pi_T].$$

This notation comes from a “Mackey decomposition”.

Combining Theorem 0.9 with 0.12, we know that $R^\tau(LT)$ is isomorphic to $K_T^{\tau+\dim(T)}(T)$.

Freed, Hopkins and Teleman constructed an isomorphism

$$FHT_T : R^\tau(LT) \rightarrow K_T^{\tau+dim(T)}(T)$$

by use of the “Dirac family”.

To explain it, let us explain “local equivalence” of “groupoids”. We start from a classical isomorphism for un-twisted equivariant K -theory. Let G act on X , N be a normal subgroup of G , and N act on X freely. Then we have an isomorphism $K_G(X) \cong K_{G/N}(X/N)$ ([Segal] Proposition 2.1). In [FHT1], they dealt with such a phenomenon more generally. They defined twisted K -theory for groupoids. A groupoid is a generalization of a space equipped with an action of a Lie group. $S//G$ is a groupoid defined by S which G acts on. For example, $K(S//G) \cong K_G(S)$. They verified that twisted K -theory is a local equivalence invariant. Local equivalence is an equivalence relation between two groupoids. In fact, if two groupoids $S_1//G_1$ and $S_2//G_2$ are locally equivalent, two orbit spaces S_1/G_1 and S_2/G_2 are homeomorphic. For example, $X//G$ and $(X/N)/(G/N)$ are locally equivalent in the above situation. By this construction, infinite dimensional spaces or Lie groups can be analyzed in some cases.

In fact, they constructed a local equivalence $hol : \mathcal{A}_{S^1 \times T} // LT \rightarrow T//T$, the holonomy map, where $\mathcal{A}_{S^1 \times T}$ is the set of connections over the trivial bundle $S^1 \times T$. Let us recall that LT acts on $\mathcal{A}_{S^1 \times T}$ by the gauge transformation. Passing through this local equivalence, elements of $K_T^{\tau+k}(T)$ can be obtained from LT -equivariant families of Fredholm operators parametrized by $\mathcal{A}_{S^1 \times T}$. If we have an irreducible positive energy representation of LT , we obtain a “Dirac family” which is a family of Fredholm operators parametrized by $\mathcal{A}_{S^1 \times T}$ ([FHT2]). Since positive energy representations are projective, we need to twist equivariant K -theory. By extending linearly, we obtain a group homomorphism

$$FHT_T : R^\tau(LT) \rightarrow K_T^{\tau+dim(T)}(T).$$

In [FHT2], they verified that FHT_T is an isomorphism by use of an explicit description of the both sides, Theorem 0.9 and Theorem 0.12.

Let us compare the common right hand side $\mathbb{Z}[\Lambda_T^\tau/\Pi_T]$ of Theorem 0.9, 0.12 with $R(T) \cong \mathbb{Z}[\Lambda_T]$, where we can see algebraic similarity between them. So we can define an analogue of the induced homomorphism of representation ring of tori.

Firstly, we rewrite induced homomorphisms of representation ring of tori. Let T and T' be tori and $f : T' \rightarrow T$ be a group homomorphism. Λ_T and $\Lambda_{T'}$ are the character groups of T and T' respectively. Representation rings of T and T' are isomorphic to the group rings of Λ_T and $\Lambda_{T'}$ respectively. That is, $R(T) \cong \mathbb{Z}[\Lambda_T]$ and $R(T') \cong \mathbb{Z}[\Lambda_{T'}]$. $f^* : R(T') \rightarrow R(T)$ is defined by the pull back of representations. When we regard Λ_T as $\text{Hom}(\Pi_T, \mathbb{Z})$, $f^*(\lambda)$ corresponds to ${}^t df(\lambda)$ ($\lambda \in \Lambda_{T'}$). That is, if we have a representation $\rho : T \rightarrow U(n)$, we can

decompose it as $\bigoplus_{i=1}^n \lambda_i$ for some $\lambda_i \in \Lambda_T$, and its image under f^* is given by

$$f^*(\rho) = f^*\left(\bigoplus_{i=1}^n \lambda_i\right) = \sum_{i=1}^n f^*(\lambda_i) = \sum_{i=1}^n {}^t df(\lambda_i).$$

In other words, if we regard $R(T)$ as ordinary K -group $K(\Lambda_T)$, f^* corresponds to $({}^t df)_!$, the push-forward, where Λ_T has the discrete topology.

Let us define the induced homomorphism $char(f) : \mathbb{Z}[\Lambda_T^\tau / \Pi_T] \rightarrow \mathbb{Z}[\Lambda_{T'}^{f^*\tau} / \Pi_{T'}]$.

Definition 0.13. When $f^*\tau$ is also positive, we define $char(f)([\lambda]_T)$ by

$$char(f)([\lambda]_T) := \sum_{i=1}^k [\mu_i]_{T'},$$

where ${}^t df(\Pi_T \cdot \lambda) = \coprod_{i=1}^k \Pi_{T'} \cdot \mu_i$, $[\lambda]_T$ is the Π_T -orbit including $\lambda \in \Lambda_T^\tau$ and $[\mu]_{T'}$ is the $\Pi_{T'}$ -orbit including $\mu \in \Lambda_{T'}^{f^*\tau}$.

Remark 0.14. Well-definedness is verified in Section 3.

If we introduce notations $char(T, \tau) := \mathbb{Z}[\Lambda_T^\tau / \Pi_T]$ and $char(T', f^*\tau) := \mathbb{Z}[\Lambda_{T'}^{f^*\tau} / \Pi_{T'}]$, $char(f)$ is a homomorphism from $char(T, \tau)$ to $char(T', f^*\tau)$. The correspondence $char$ seems to be a functor. However, it is not true. That is, there exist homomorphisms $T \xrightarrow{f} T' \xrightarrow{g} T''$ such that $char(g \circ f) \neq char(f) \circ char(g)$. We call such a correspondence between two categories a “quasi functor”.

Definition 0.15. Let \mathcal{C}_1 and \mathcal{C}_2 be categories. $F : \mathcal{C}_1 \rightarrow \mathcal{C}_2$ is a quasi functor if the following data are given.

- (1) For any $O \in \text{Obj}(\mathcal{C}_1)$, $F(O) \in \text{Obj}(\mathcal{C}_2)$.
- (2) For any $f \in \text{Mor}_{\mathcal{C}_1}(O_1, O_2)$, $F(f) \in \text{Mor}_{\mathcal{C}_2}(F(O_2), F(O_1))$ satisfying $F(id) = id$.

That is, a quasi functor is a “functor” which is allowed $F(f_1 \circ f_2) \neq F(f_2) \circ F(f_1)$.

However, the quasi functor $char$ holds functoriality we need. We verify the following theorem (Theorem 3.10).

Theorem 0.16. For any local injection $f : T' \rightarrow T$ and positive central extension τ of LT , we can define the “orthogonal completion torus” $(T' / \ker(f))^\perp \subseteq T$ satisfying the following condition. Let $j : (T' / \ker(f))^\perp \hookrightarrow T$ be the natural inclusion. $f \cdot j : T' / \ker(f) \times (T' / \ker(f))^\perp \rightarrow T$ is defined by $f \cdot j(t_1, t_2) = f(t_1)j(t_2)$.

- (1) $f \cdot j : T' / \ker(f) \times (T' / \ker(f))^\perp \rightarrow T$ is a finite covering.
- (2) $(f \cdot j)^*\tau$ can be written as $p_1^*\tau_1 + p_2^*\tau_2$, where τ_1, τ_2 are positive central extensions of $L(T' / \ker(f))$ and $L(T' / \ker(f))^\perp$ respectively.

Moreover, we can decompose f as follows.

$$T' \xrightarrow{q} T' / \ker(f) \xrightarrow{i_1} T' / \ker(f) \times (T' / \ker(f))^\perp \xrightarrow{f \cdot j} T,$$

where i_1 is the natural inclusion into the first factor.

“Functoriality we need” is that $\text{char}(f) = \text{char}(q) \circ \text{char}(i_1) \circ \text{char}(f \cdot j)$ (Theorem 3.12). From this theorem, it is sufficient to deal with only cases of direct products or finite coverings.

This paper consists of 5 sections.

In Section 1, we describe known results and definitions about twisted equivariant K -theory and positive energy representation of loop groups.

In Section 2, we define some categories and quasi functors and rewrite FHT isomorphism in our terminology. Then we verify our main theorem under the assumption that Theorem 0.6 and 0.7 hold.

In Section 3, we study the quasi functor char . We define the induced homomorphism $\text{char}(f)$, which is an analogue of pull back of representations of tori. The quasi functor char is the most useful for computation. We deal with not only tori but also compact connected Lie groups with torsion-free π_1 .

In Section 4, we study the quasi functor $t.e.K$, a modification of twisted equivariant K -theory. We define $f^\#$ so that it holds naturality with $\text{char}(f)$. We deal with not only tori but also compact connected Lie groups with torsion-free π_1 .

In Section 5, we study the quasi functor RL , a modification of positive energy representation group of loop groups. We define $f^!$ so that it holds naturality with $\text{char}(f)$.

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1 General definitions and known results

1.1 Loop groups and their positive energy representations

Representation group of LT at level τ is one of our main objects, where T is a torus. In this section, we deal with not only tori but also general Lie group with torsion-free π_1 . Firstly, we explain central extensions of loop groups.

1.1.1 Definitions and some formulae about central extension

Let G be a compact connected Lie group with torsion-free π_1 . Let LG be the group of smooth maps from S^1 to G , loop group of G . Its topology is defined so that l_n converges to l if and only if all derivatives $\frac{d^k l_n}{d\theta^k}$ converges to $\frac{d^k l}{d\theta^k}$ uniformly. This is one of the simplest infinite dimensional Fréchet Lie groups.

S^1 acts on LG via transformation of parameter, that is, $(\theta_0^* l)(\theta) := l(\theta - \theta_0)$ ($l \in LG$). When we make S^1 act on loop groups, we write S^1 as \mathbb{T}_{rot} . So we can define the new Lie group $LG \rtimes \mathbb{T}_{rot}$.

The Lie algebra of LG is the loop algebra $L\mathfrak{g} := C^\infty(S^1, \mathfrak{g})$. Its addition, scalar multiplication and Lie bracket are induced by ones of \mathfrak{g} . We write the Lie algebra of $LG \rtimes \mathbb{T}_{rot}$ as $L\mathfrak{g} \oplus i\mathbb{R}_{rot}$. If d is an infinitesimal generator of the action of \mathbb{T}_{rot} , $[d, \beta] = \frac{d\beta}{d\theta}$ for $\beta \in L\mathfrak{g}$.

We need the following definition in order to define positive energy representations of loop groups, because positive energy representations must be projective unless it is trivial.

Definition 1.1 ([FHT2]). *A central extension*

$$1 \rightarrow U(1) \xrightarrow{i} LG^\tau \xrightarrow{p} LG \rightarrow 1$$

is admissible if

(1) The action of \mathbb{T}_{rot} lifts to LG^τ . Therefore, we can define the new Lie group $LG^\tau \rtimes \mathbb{T}_{rot}$.

(2) $LG^\tau \rtimes \mathbb{T}_{rot}$ acts on $L\mathfrak{g}^\tau \oplus i\mathbb{R}_{rot}$ via adjoint action. There exists an $LG^\tau \rtimes \mathbb{T}_{rot}$ -invariant symmetric bilinear form $\ll \cdot, \cdot \gg_\tau$ on $L\mathfrak{g}^\tau \oplus i\mathbb{R}_{rot}$ such that $\ll K, d \gg_\tau = -1$, where K is the infinitesimal generator of $i(U(1))$. If the restriction of $\ll \cdot, \cdot \gg_\tau$ to $L\mathfrak{g}$ is positive definite, τ is called a positive central extension.

Remark 1.2. Bilinear form $\ll \cdot, \cdot \gg_\tau$ determines a splitting $L\mathfrak{g} \rightarrow L\mathfrak{g}^\tau \oplus i\mathbb{R}_{rot}$. So we can “restrict” $\ll \cdot, \cdot \gg_\tau$ to $L\mathfrak{g}$ ([FHT2] Lemma 2.18).

Definition 1.3. Let $f : H \rightarrow G$ be a smooth group homomorphism, then the homomorphism $Lf : LH \rightarrow LG$ is defined by $l \mapsto f \circ l$. We can pull back the central extension τ of LG along Lf and we write it as $f^*\tau$. That is, the following commutative diagram holds.

$$\begin{array}{ccccc} U(1) & \xrightarrow{i} & LG^\tau & \xrightarrow{p} & LG \\ \parallel & & \uparrow \widetilde{Lf} & & \uparrow Lf \\ U(1) & \xrightarrow{i} & LH^{f^*\tau} & \xrightarrow{p} & LH \end{array}$$

Lemma 1.4. If τ is admissible, so is $f^*\tau$.

Proof. Let us notice that \widetilde{Lf} can be extended to $LH^{f^*\tau} \rtimes \mathbb{T}_{rot}$ by $\widetilde{Lf}|_{\mathbb{T}_{rot}} := id_{\mathbb{T}_{rot}}$. We use the same character for this extension.

(1) is clear from the definition.

(2) is verified by defining the bilinear form on $L\mathfrak{g}^\tau \oplus i\mathbb{R}_{rot}$ explicitly as follows. $d\widetilde{Lf}$ is the tangent map of \widetilde{Lf} .

$$\ll v, w \gg_{f^*\tau} := \ll d\widetilde{Lf}(v), d\widetilde{Lf}(w) \gg_\tau.$$

Since $\widetilde{Lf}|_{U(1)}$ is the identity, $d\widetilde{Lf}(K) = K$. Moreover, $d\widetilde{Lf}(d) = d$ by the definition. Therefore, the above condition is satisfied. \square

Definition 1.5. Let τ_1 and τ_2 be admissible central extensions of LG . Then we can define the tensor product between them by

$$LG^{\tau_1+\tau_2} := LG^{\tau_1} \otimes LG^{\tau_2} = LG^{\tau_1} \times LG^{\tau_2} / LG.$$

Lemma 1.6 ([FHT2] Section 2.2). If τ_1 and τ_2 are admissible central extensions of LG , so is $\tau_1 + \tau_2$. Invariant bilinear form can be defined by

$$\ll \cdot, \cdot \gg_{\tau_1+\tau_2} = \ll \cdot, \cdot \gg_{\tau_1} + \ll \cdot, \cdot \gg_{\tau_2}.$$

Remark 1.7. Lie algebra $L\mathfrak{g}^\tau \oplus i\mathbb{R}_{\text{rot}}$ for simple \mathfrak{g} and the “universal central extension” τ can be thought as a “completion” of the affine algebra $\widehat{\mathfrak{g}}$. From this view point, bilinear form $\ll \cdot, \cdot \gg_\tau$ can be thought as a lift of Killing form of \mathfrak{g} to $\widehat{\mathfrak{g}}$.

When G is a torus T , admissibility implies the following lemma. However, we need some terminologies about tori or their representations.

$\Pi_T := \ker(\exp : \mathfrak{t} \rightarrow T) \cong \pi_1(T) \cong H_1(T)$ is a lattice in \mathfrak{t} .

$\Lambda_T := \text{Hom}(\Pi_T, \mathbb{Z}) \cong \text{Hom}(T, U(1)) \cong H^1(T)$ is the character group. We can regard it as the set of irreducible representations of T from Schur’s lemma (any irreducible representation of commutative group is 1 dimensional).

Lemma 1.8 ([FHT2] Proposition 2.27). LT has a canonical decomposition $LT \cong T \times \Pi_T \times U$, where T is the set of initial values of loops, Π_T is the set of “rotation numbers” (naturally isomorphic to $\pi_1(T) \cong \pi_0(LT)$) and

$$U := \exp \left\{ \beta : S^1 \rightarrow \mathfrak{t} \left| \int_{S^1} \beta(s) ds = 0 \right. \right\}$$

is the set of derivatives of contractible loops whose initial values are 0.

Let τ be an admissible central extension of LT , then the above decomposition is inherited partially, that is $LT^\tau \cong (T \times \Pi)^\tau \otimes U^\tau$.

This lemma allows us to define κ^τ and describe some formulae. κ^τ is the tangent map of $\widetilde{\kappa}^\tau$ defined below.

Definition 1.9 ([FHT2] Proposition 2.27). A homomorphism $\widetilde{\kappa}^\tau : \Pi_T \rightarrow \text{Hom}(T, U(1))$ is defined by

$$\widetilde{\kappa}^\tau(X)(t) := \widetilde{\phi}_X t \widetilde{\phi}_X^{-1} \widetilde{t}^{-1} \in i(U(1)),$$

where \widetilde{t} is a chosen lift of $t \in T$, and $\widetilde{\phi}_X$ is a chosen lift of $\phi_X \in LT$ which is a geodesic loop corresponding to $X \in \Pi_T \cong \pi_1(T)$ whose initial value is 0. This definition is independent of the choices of lifts because they are determined up to $i(U(1))$ which is in the center of LT^τ .

This definition means that κ^τ represents how non-commutativity is caused by the central extension.

Lemma 1.10 ([FHT2] Proposition 2.27). $\widetilde{\kappa}^\tau$ determines a symmetric bilinear form $\langle n, m \rangle_\tau = -\frac{1}{\sqrt{-1}} \widetilde{\kappa}^\tau(n)(m)$ on Π_T for $n, m \in \Pi_T$.
Moreover, when we extend $\langle \cdot, \cdot \rangle_\tau$ to \mathfrak{t} , $\langle v, w \rangle_\tau = \ll v, w \gg_\tau$.

Proposition 1.11. If a central extension τ of LT is positive, so is $\langle \cdot, \cdot \rangle_\tau$.

We can verify the following formulae.

Let $f : T' \rightarrow T$ be a local injection, that is, the tangent map df is injective. Then we have the associated homomorphism $Lf : LT' \rightarrow LT$ and $\widetilde{Lf} : LT'^{f*\tau} \rightarrow LT^\tau$.

Lemma 1.12.

$$\kappa^{f*\tau} = {}^t df \circ \kappa^\tau \circ df$$

Proof. Let $X \in \Pi_{T'}$, $t \in T'$. Take lifts $\widetilde{\phi}_X \in LT'^{f*\tau}$ and $\widetilde{t} \in LT'^{f*\tau}$ of $\phi_X \in LT'$ and t . Let us notice that lifts of $\phi_{df(X)}$ and $f(t)$ can be chosen by $\widetilde{Lf}(\widetilde{\phi}_X)$ and $\widetilde{Lf}(\widetilde{t})$ respectively. By the definition, $\widetilde{Lf}|_{i(U(1))}$ is the identity. Therefore, we can compute $\widetilde{\kappa}^{f*\tau}_X(t)$ using $\widetilde{\kappa}^\tau$ as follows.

$$\begin{aligned} \widetilde{\kappa}^{f*\tau}_X(t) &= \widetilde{Lf}(\widetilde{\kappa}^{f*\tau}_X(t)) \\ &= \widetilde{Lf}(\widetilde{\phi}_X \widetilde{\phi}_X^{-1} \widetilde{t}^{-1}) = \widetilde{Lf}(\widetilde{\phi}_X) \widetilde{Lf}(\widetilde{t}) \widetilde{Lf}(\widetilde{\phi}_X^{-1}) \widetilde{Lf}(\widetilde{t}^{-1}) \\ &= \widetilde{\phi_{df(X)}} \widetilde{f(t)} \widetilde{\phi_{df(X)}^{-1}} \widetilde{f(t)^{-1}} = \widetilde{\kappa}^\tau_{df(X)}(f(t)) \end{aligned}$$

By taking the tangent map, we obtain the conclusion. \square

From this lemma, we can verify the following one.

Let T_1 and T_2 be tori, $T := T_1 \times T_2$, $i_j : T_j \rightarrow T$ be the natural inclusion into the j 'th factor, $p_j : T \rightarrow T_j$ be the projection onto the j 'th factor and τ_j be an admissible central extension of LT_j ($j = 1, 2$). Then $\tau = p_1^* \tau_1 + p_2^* \tau_2$ is an admissible central extension of LT .

In this case we have canonical isomorphisms $LT \cong LT_1 \times LT_2$ and $LT^\tau \cong LT_1^{\tau_1} \times LT_2^{\tau_2} / U(1)$. Therefore, for any $l_1, l'_1 \in LT_1^{\tau_1}$, $l_2, l'_2 \in LT_2^{\tau_2}$, $l_1 \otimes l_2 \cdot l'_1 \otimes l'_2 = l_1 l'_1 \otimes l_2 l'_2$. Moreover, $LT^\tau \cong U_1^{\tau_1} \otimes U_2^{\tau_2} \otimes (T_1 \times \Pi_{T_1})^{\tau_1} \otimes (T_2 \times \Pi_{T_2})^{\tau_2}$.

Lemma 1.13. $\kappa^\tau : \Pi_{T_1 \times T_2} \rightarrow \Lambda_{T_1 \times T_2}$ is the composition of

$$\Pi_{T_1 \times T_2} \xrightarrow{dp_1 \oplus dp_2} \Pi_{T_1} \oplus \Pi_{T_2} \xrightarrow{\kappa^{\tau_1} \oplus \kappa^{\tau_2}} \Lambda_{T_1} \oplus \Lambda_{T_2} \xrightarrow{{}^t dp_1 \oplus {}^t dp_2} \Lambda_{T_1 \times T_2}.$$

Formally

$$\kappa^\tau = \begin{pmatrix} \kappa^{\tau_1} & 0 \\ 0 & \kappa^{\tau_2} \end{pmatrix}$$

We often use these formulae in this paper.

1.1.2 Positive energy representation

We can define positive energy representations, which is one of the main objects in this paper.

Let $U(1) \xrightarrow{i} LG^\tau \xrightarrow{p} LG$ be an admissible central extension.

Definition 1.14 ([PS][FHT2]). *A continuous homomorphism $\rho : LG^\tau \rightarrow U(V)_{c.o.}$ is called a positive energy representation at level τ if*

- (1) $\rho \circ i(e^{2\pi\sqrt{-1}}) = e^{2\pi\sqrt{-1}}id_V$ (the definition of τ -twisted representations).
- (2) ρ lifts to a continuous homomorphism $\hat{\rho} : LG^\tau \rtimes \mathbb{T}_{rot} \rightarrow U(V)_{c.o.}$.
- (3) $E := \frac{1}{\sqrt{-1}}\hat{\rho}(d)$ which is called an energy operator is self-adjoint with discrete spectrum bounded below.

V is a separable complex Hilbert space as a representation space, $U(V)_{c.o.}$ is topologized by the compact open topology (or the strong topology) and d is the infinitesimal generator of \mathbb{T}_{rot} .

A positive energy representation is said to be finitely reducible if it is a finite sum of irreducible representations.

Isomorphism classes of finitely reducible positive energy representations at level τ form an abelian semigroup under the direct sum.

Definition 1.15. $R^\tau(LG)$ is the Grothendieck completion of this semigroup. In other words, $R^\tau(LG)$ is the \mathbb{Z} free module generated by the set of isomorphism classes of irreducible positive energy representations. We call it a representation group of LG at level τ .

Remark 1.16. In fact, it has a ring structure and it is called the “Verlinde ring”. The product is given by the fusion product of conformal field theory. However, we do not deal with the product in this paper.

Positive energy representations at level τ have been classified ([PS]).

Let G be a connected compact Lie group with torsion-free π_1 and τ be a positive central extension of LG . We fix a maximal torus T of G .

We need to introduce some terminologies to describe the following theorems.

We can define central extensions G^τ and T^τ by the restrictions of LG^τ to G and T . That is, the commutative following diagram holds.

$$\begin{array}{ccccccc}
 1 & \longrightarrow & U(1) & \xrightarrow{i} & T^\tau & \xrightarrow{p} & T \longrightarrow 1 \\
 \parallel & & \parallel & & \downarrow & & \downarrow \\
 1 & \longrightarrow & U(1) & \xrightarrow{i} & G^\tau & \xrightarrow{p} & G \longrightarrow 1 \\
 \parallel & & \parallel & & \downarrow & & \downarrow \\
 1 & \longrightarrow & U(1) & \xrightarrow{i} & LG^\tau & \xrightarrow{p} & LG \longrightarrow 1
 \end{array}$$

We can verify that T^τ is a maximal torus of G^τ .

So we can define a τ -twisted representation of T^τ . A τ -twisted representation is a homomorphism $\rho : T^\tau \rightarrow U(V)$ such that $\rho \circ i(e^{\sqrt{-1}\theta}) = e^{\sqrt{-1}\theta}id_V$, where V

is a representation space. Λ_T^τ is the set of irreducible τ -twisted representations. It can be thought as the set of $\lambda : \Pi_{T^\tau} \rightarrow \mathbb{Z}$ such that the composition of $\mathbb{Z} = \Pi_{U(1)} \xrightarrow{di} \Pi_{T^\tau} \xrightarrow{\lambda} \mathbb{Z}$ is identity. We can define an action of Λ_T on Λ_T^τ by the tensor product of the representations. We write it $\lambda + \lambda'$ ($\lambda \in \Lambda_T, \lambda' \in \Lambda_T^\tau$). The tensor product of the representations is defined by

$$(\lambda + \lambda')(t) = \lambda(p(t)) \cdot \lambda'(t),$$

where $t \in T^\tau$. Since $p \circ i(z) = 1$, $\lambda + \lambda'$ is also a τ -twisted representation.

The Weyl group of G and G^τ are naturally isomorphic. We write them $W(G)$. Since $W(G)$ acts on Π_T , we can define the semi direct product

$$W_{aff}^e(G) := \Pi_T \rtimes W(G),$$

we call it the extended affine Weyl group. It is the Weyl group of LG^τ , whose maximal torus is T^τ . Since $W(G)$ acts on Π_{T^τ} , $W(G)$ acts on Λ_T^τ . And Π_T acts on Λ_T^τ through the homomorphism κ^τ and $\Lambda_T \hookrightarrow \Lambda_T^\tau$. Therefore, $W_{aff}^e(G)$ acts on Λ_T^τ . We write the orbit space $\Lambda_T^\tau / \kappa^\tau(W_{aff}^e(G))$.

$\mathbb{Z}[(\Lambda_T^\tau / \kappa^\tau(W_{aff}^e(G)))_{reg}]$ is the free \mathbb{Z} module generated by the set of $W_{aff}^e(G)$ -regular orbits $\Lambda_T^\tau / \kappa^\tau(W_{aff}^e(G))$. A regular orbit is an orbit whose stabilizer is trivial.

Let σ be the spin extension of LG defined in [FHT2]. It is defined by the irreducible Clifford module of $L\mathfrak{g}^*$ and the coadjoint action of LG on $L\mathfrak{g}^*$. Let us suppose that $\tau - \sigma$ is positive.

Lemma 1.17 ([Adams]). *The map*

$$\rho - shift : \Lambda_T^{\tau-\sigma} / \kappa^{\tau-\sigma}(W_{aff}^e(G)) \xrightarrow{\cong} (\Lambda_T^\tau / \kappa^\tau(W_{aff}^e(G)))_{reg}$$

defined by $[\lambda] \mapsto [\lambda + \rho]$ is bijective, where ρ is the half of the sum of the positive roots of G .

Remark 1.18. *When G is a torus T , $W_{aff}^e(T) = \Pi_T$ and $\rho = 0$. Since any orbit is regular, the above lemma is trivial.*

Theorem 1.19 ([FHT2]). *Isomorphism classes of positive energy $(\tau - \sigma)$ -twisted representations of $LG^{\tau-\sigma}$ are parametrized by the $W_{aff}^e(G)$ -orbits in $\Lambda_T^{\tau-\sigma}$ by taking the lowest weight. From Lemma 1.17, we have the following isomorphisms*

$$R^{\tau-\sigma}(LG) \xrightarrow{L.W.G} \mathbb{Z}[\Lambda_T^{\tau-\sigma} / W_{aff}^e(G)] \xrightarrow{\rho - shift} \mathbb{Z}[(\Lambda_T^\tau / W_{aff}^e(G))_{reg}].$$

$L.W.G$ is defined as follows. If V is an irreducible positive energy representation of LG at level $\tau - \sigma$, $L.W.G(V)$ is the lowest weight of V .

Definition 1.20.

$$l.w.G := \rho - shift \circ L.W.G$$

Since $W_{aff}^e(G) = \Pi_T \rtimes W(G)$ is the Weyl group of LG or $LG^{\tau-\sigma}$, this theorem is an analogue of the Cartan Weyl's highest weight theory for compact Lie groups.

1.2 Twisted equivariant K -theory

1.2.1 The definitions and the classification of twistings

Let X be a manifold, P be a projective bundle. We can define twisted K -theory associated with it following [AS]. We do not deal with non-trivial gradings, that is, projective bundle P has two disjoint sub projective bundles P_1 and P_2 such that $\mathcal{H}|_x = \mathcal{H}_1|_x \oplus \mathcal{H}_2|_x$, where $\mathcal{H}_j|_x$ is a chosen lift of $P_j|_x$ ($j = 1, 2, x \in X$). This assumption is automatically satisfied for the space whose first cohomology group with \mathbb{Z}_2 coefficient is zero. An isomorphism class of projective bundles is called a twisting. We define the classifying bundle $Fred^{(0)}(P)$ for twisted K -theory. Firstly, we define the space of Fredholm operators of a projective space.

Definition 1.21. $Fred^{(0)}(\mathbb{P}(\mathcal{H}))$ is the space of odd skew-adjoint Fredholm operators $A : \mathcal{H} \rightarrow \mathcal{H}$, for which $A^2 + 1$ is a compact operator. It is topologized so that $A \mapsto (A, A^2 + 1) \in B(\mathcal{H})_{c.o.} \times K(\mathcal{H})_{norm}$ is continuous.

Remark 1.22. (1) If we take an another lift \mathcal{K} of $\mathbb{P}(\mathcal{H})$, we can take the unique isomorphism $\phi : \mathcal{H} \rightarrow \mathcal{K}$ up to $U(1)$ action. ϕ makes the following diagram commute.

$$\begin{array}{ccc} \mathcal{H} & \xrightarrow{\phi} & \mathcal{K} \\ & \searrow & \swarrow \\ & P(\mathcal{H}) & \end{array}$$

So if $A : \mathcal{H} \rightarrow \mathcal{H}$ is a linear map, $\phi \circ A \circ \phi^{-1} : \mathcal{K} \rightarrow \mathcal{K}$ is independent of the choice of ϕ , so the above is well-defined.

(2) For the same reason we can take tensor products $P_1 \hat{\otimes} P_2$, $P \hat{\otimes} \mathcal{L}$ and so on, where \mathcal{L} is a Hilbert bundle (can be finite rank). That is, $P_1 \hat{\otimes} P_2|_x := \mathbb{P}(\mathcal{H}_1|_x \hat{\otimes} \mathcal{H}_2|_x)$ and $P \hat{\otimes} \mathcal{L}|_x := \mathbb{P}(\mathcal{H}|_x \hat{\otimes} \mathcal{L}|_x)$, where $\mathcal{H}|_x$, $\mathcal{H}_1|_x$ and $\mathcal{H}_2|_x$ are chosen lifts of P , P_1 and P_2 respectively. $\mathbb{P}(V)$ is the projectification of the vector space V .

Let us define twisted K -theory. Let X be a manifold, X' be a closed subset of X .

Definition 1.23. Let P be a projective bundle. The classifying bundle is defined by

$$Fred^{(0)}(P) := \bigcup_{x \in X} Fred^{(0)}(P_x \otimes l^2).$$

We write the space of sections as

$$\Gamma_c(X, X', Fred^{(0)}(P))$$

$$:= \{s : X \rightarrow Fred^{(0)}(P) | \text{continuous section with compact support and } \text{supp}(s) \cap X' = \emptyset\},$$

where $\text{supp}(s) := \{x \in X | s_x \text{ is not invertible}\} = \{x \in X | \ker(s_x) \neq 0\}$ is called the support of s . We define twisted K -theory as a generalized cohomology theory.

$$K_P^0(X, X') := \pi_0(\Gamma_c(X, X', \text{Fred}^{(0)}(P))).$$

For $n \neq 0$,

$$K_P^n(X, X') := \begin{cases} K_P^0(X \times [0, 1]^{-n}, X \times \partial[0, 1]^{-n} \cup X' \times [0, 1]^{-n}) & n < 0 \\ K_P^{-n}(X, X') & n > 0 \end{cases}$$

Remark 1.24. The functor K_P^n has the classifying bundle whose fiber is the space of Fredholm operators satisfying some conditions of a projective space ([FHT1]). Therefore, we regard an element of $K_P^n(X)$ as a homotopy class of a family of Fredholm operators.

Let X and Y be smooth manifolds, X' and Y' be closed subset of X and Y respectively, $F : Y \rightarrow X$ be a smooth proper map such that $F(X') \subseteq Y'$, and P be a projective bundle over X .

Definition 1.25.

$$F^* : K_P^k(X, X') \rightarrow K_{F^*P}^k(Y, Y')$$

is defined by $F^*([s]) := [s \circ F]$, where $[\cdot]$ means a homotopy class of sections.

Theorem 1.26 ([AS]). Twisted K -theory is a two periodic generalized cohomology theory and. That is, $K_P^k(X) \cong K_P^{k+2}(X)$ (Bott periodicity).

Twistings are classified in [AS]

Proposition 1.27 ([AS] Proposition 2.1, 2.2). (1) Isomorphism classes of twistings are classified by $H^3(X, \mathbb{Z})$ by taking Dixmier-Douady class.

(2) The group of connected components of the group of automorphisms of P is isomorphic to $H^2(X, \mathbb{Z})$ which classifies isomorphism classes of complex line bundles.

(3) For $l \in H^2(X, \mathbb{Z})$, the action of l on $K_P^k(X)$ is given by the tensor product with L , where L is a complex line bundle whose first Chern class is l and we regard it as an element of $K(X)$.

Remark 1.28. Since $PU(\mathcal{H})$ is an Eilenberg MacLane space $K(\mathbb{Z}, 2)$, the classifying space of it is homotopic to $K(\mathbb{Z}, 3)$. It is the classifying space of the functor H^3 . Therefore, (1) holds. Let $\alpha : P \rightarrow P$ be an isomorphism, then $\alpha : P_x \rightarrow P_x$ has a lift $\widetilde{\alpha}_x : \mathcal{H}_x \rightarrow \mathcal{H}_x$ up to $U(1)$. Therefore, α determines a $U(1)$ principal bundle. Therefore, (2) holds. (3) tells us that $H^2(X, \mathbb{Z})$ acts on $K_P^n(X)$ non-trivially.

Example 1.29. Let X be a compact manifold. Let $p : E \rightarrow X$ be an oriented real vector bundle whose rank is k over X . Let us take a local trivialization $\{E|_{U_\alpha} \cong U_\alpha \times \mathbb{R}^k\}_\alpha$. Since any oriented vector space V has the unique irreducible Clifford module Δ_V ([Furuta]), we can define a “local Spinor bundle”. That is,

$S|_{U_\alpha} := U_\alpha \times \Delta_{\mathbb{R}^k}$. From Schur's lemma, we can take a transition function $\phi_{\alpha\beta} : U_\alpha \cap U_\beta \rightarrow GL(\Delta_{\mathbb{R}^k})$ up to \mathbb{C}^* . Therefore, the family $\{\mathbb{P}(S|_{U_\alpha})\}_\alpha$ defines a projective bundle whose fiber is $\mathbb{P}(\Delta_{\mathbb{R}^k})$. Dixmier-Douady class of this bundle is the image of $w_2(E)$ under the Bockstein operator $\beta : H^2(X, \mathbb{Z}_2) \rightarrow H^3(X, \mathbb{Z})$. This is the obstruction for existence of a Spin^c structure. $\beta(w_2(E))$ -twisted K -theory is studied in [Karoubi] and [Furuta]. $K^{\beta(w_2(E))}(X)$ is written by $K^E(X)$ in [Karoubi], $K(X, Cl(E))$ in [Furuta]. Thom isomorphism states that a natural homomorphism $K(X) \rightarrow K^{\beta(w_2(p^*E))}(E)$ is isomorphic.

Twisted K -theory has an equivariant version just like ordinary case. Let G be a compact Lie group and act on X smoothly. We explain only the changes or additional points from the non-equivariant cases. Let X' be a closed G -invariant subset.

Definition 1.30. Let P be a G -equivariant projective bundle. An isomorphism class of G -equivariant projective bundles is called a G -equivariant twisting. Then the classifying bundle of twisted equivariant K -theory is defined by

$$\text{Fred}_G^{(0)}(P) := \bigcup_{x \in X} \text{Fred}^{(0)}(P_x \otimes L^2(G) \otimes l^2).$$

Since G acts on $L^2(G)$ via the left regular representation and on l^2 trivially, we can define the space of G -equivariant sections.

$$\Gamma_{c,G}(X, X', \text{Fred}_G^{(0)}(P))$$

$$:= \{s : X \rightarrow \text{Fred}_G^{(0)}(P) \mid \text{continuous, equivariant, compactly supported and } \text{supp}(s) \cap X' = \emptyset\}.$$

We define twisted equivariant K -theory as a generalized cohomology theory for G -spaces.

$$K_{G,P}^0(X, X') := \pi_0(\Gamma_{c,G}(X, X', \text{Fred}_G^{(0)}(P)))$$

$K_{G,P}^n(X, X')$ is defined just like the non-equivariant cases for $n \neq 0$. “Compactly supported” means that the quotient space $\text{supp}(s)/G$ is compact.

Bott periodicity remains valid.

Remark 1.31. Any finite dimensional representation space of G can be embedded into $L^2(G) \otimes l^2$ from Peter-Weyl theorem.

Let X be a smooth manifold and G act on X smoothly. Let $f : H \rightarrow G$ be a smooth group homomorphism. Let P be a G -equivariant projective bundle over X , which is also H -equivariant through f . When we regard P as an H -equivariant one, we write it f^*P .

Definition 1.32.

$$f^* : K_{G,P}^k(X) \rightarrow K_{H,f^*P}^k(X)$$

is defined by regarding G -equivariant class $[s]$ as an H -equivariant one through f .

Classification of G -equivariant twistings has been known.

Proposition 1.33 ([AS] Proposition 6.3). *(1) Isomorphism classes of twistings are classified by $H_G^3(X, \mathbb{Z})$ by taking Dixmier-Douady class.*

(2) The group of connected components of the group of automorphisms of P is isomorphic to $H_G^2(X, \mathbb{Z})$ which classifies isomorphism classes of G -equivariant complex line bundles.

(3) For $l \in H_G^2(X, \mathbb{Z})$, the action of l on $K_{G,P}^k(X)$ is given by the tensor product with L , where L is a complex line bundle whose first equivariant Chern class is l and we regard L as an element of $K_G(X)$.

Just like non-equivariant cases, $H_G^2(X)$ acts on $K_{G,P}^k(X)$ non-trivially.

G -equivariant twisting over G we deal with in this paper can be obtained by the following lemma, where G acts on itself by conjugation.

From now on, we write $K_{G,P}^k(X)$ as $K_G^{\tau+k}(X)$, where τ is Dixmier-Douady class of a projective bundle P .

Lemma 1.34 ([AS]). *If a central extension LG^τ has a τ -twisted representation, τ gives a G -equivariant twisting over G .*

Proof. Firstly, we explain the relationship between projective representations of LG and central extensions of LG .

Let us assume that LG has a projective representation $\rho : LG \rightarrow PU(V)$. From the central extension $U(1) \rightarrow U(V) \rightarrow PU(V)$, we can define a central extension $LG^\tau = LG \times_{PU(V)} U(V)$ and a unitary representation $\hat{\rho} : LG^\tau \rightarrow U(V)$ such that $\hat{\rho} \circ i(e^{\sqrt{-1}\theta}) = e^{\sqrt{-1}\theta} id_V$ as follows.

$$\begin{array}{ccccc} U(1) & \longrightarrow & U(V) & \longrightarrow & PU(V) \\ \parallel & & \uparrow \hat{\rho} & & \uparrow \rho \\ U(1) & \xrightarrow{i} & LG^\tau & \xrightarrow{p} & LG \end{array}$$

In this situation, $(\hat{\rho}, V)$ is called a τ -twisted representation.

If a central extension $U(1) \xrightarrow{i} LG^\tau \xrightarrow{p} LG$ is given, a representation

$$\hat{\rho} : LG^\tau \rightarrow U(V)$$

is also called a τ -twisted representation if $\hat{\rho} \circ i(e^{\sqrt{-1}\theta}) = e^{\sqrt{-1}\theta} id_V$. In this case, we can define a projective representation $\rho : LG \rightarrow PU(V)$.

Two definitions coincide.

Let $\rho : LG \rightarrow PU(V)$ be a projective representation. Then LG acts on $\mathbb{P}(V \otimes l^2 \otimes L^2(G))$.

We have a G -equivariant LG principal bundle $LG \rightarrow PG \rightarrow G$, where G acts on PG by left multiplication, and on itself by conjugation. PG is the path space of G defined by

$$PG := \{p : \mathbb{R} \rightarrow G | p(\theta + 2\pi)p(\theta)^{-1} \in G \text{ is independent of } \theta\}.$$

Then we can define a G -equivariant projective bundle

$$PG \times_{LG} \mathbb{P}(V \otimes l^2 \otimes L^2(G)).$$

□

Lastly, let us study the push-forward maps. When Y is a compact manifold and X is a manifold, any smooth map $F : Y \rightarrow X$ can be written as a composition of closed embedding $i : Y \hookrightarrow X \times S^{2N}$ and the trivial S^{2N} fibration $p : X \times S^{2N} \rightarrow X$ for sufficient large N . Let τ be a twisting over X . Since we do not deal with non-trivial gradings, we assume that the normal bundle of $i(Y)$ in $X \times S^{2N}$ is orientable. Push-forward map

$$i_! : K^{F^*\tau + W_3(F) + k + \dim X - \dim Y}(Y) \rightarrow K^{p^*\tau + k}(X \times S^{2N})$$

is defined by “zero extension” and

$$p_! : K^{p^*\tau + k}(X \times S^{2N}) \rightarrow K^{\tau + k}(X)$$

is defined by “family index theorem”, where $W_3(F)$ is the image of $w_2(X) - F^*(w_2(Y)) \in H^2(X, \mathbb{Z}_2)$ under the Bockstein homomorphism $\beta : H^2(X, \mathbb{Z}_2) \rightarrow H^3(X, \mathbb{Z})$.

Definition 1.35.

$$F_! := p_! \circ i_!$$

$F_!$ is independent of choices of i and N . One can find more information in [CW].

Remark 1.36. *We can generalize for non-compact Y because we deal with compactly supported K -theory.*

Push-forward map has functoriality. Let X, Y and Z be manifolds, τ be a twisting over Z , and $X \xrightarrow{f} Y \xrightarrow{g} Z$ be smooth maps satisfying the above assumption.

Lemma 1.37 ([CW]).

$$g_! \circ f_! = (g \circ f)_!$$

Let $p : Y \rightarrow X$ be a fiber bundle whose fiber is discrete and P be a projective bundle over X . Then, we can describe $p_! : K_{p^*P}^k(Y) \rightarrow K_P^k(X)$ as follows.

Lemma 1.38. *If s is a section of $p^*\text{Fred}(P)$, the homotopy class $[s]$ determines an element of $K^{p^*\tau}(Y)$. Then, we have an explicit formula*

$$p_!([s]) = [\{ \bigoplus_{y \in p^{-1}(x)} s_y \}_{x \in X}],$$

where we use an isomorphism $\oplus(P \otimes l^2) \cong P \otimes (\oplus l^2) \cong P \otimes l^2$ and Kuiper’s theorem.

1.2.2 Computation of $K_G^{\tau+k}(G)$

Freed, Hopkins and Teleman computed $K_G^{\tau+k}(G)$ by use of spectral sequences or a Mackey decomposition in [FHT1] or [FHT3].

Theorem 1.39 ([FHT1] [FHT3]). *Let τ be a G -equivariant twisting over G coming from a positive central extension of LG . Then the following isomorphism holds.*

$$K_G^{\tau+k}(G) = \begin{cases} 0 & (k = \text{rank}(G) + 1 \pmod{2}) \\ \mathbb{Z}[(\Lambda^\tau / \kappa^\tau(W_{\text{aff}}^e(G)))_{\text{reg}}] & (k = \text{rank}(G) \pmod{2}) \end{cases}$$

2 Three contravariant quasi functors and FHT isomorphisms

As a preliminary, we introduce some categories and quasi functors. Then we rewrite FHT isomorphisms using our terminologies and verify Theorem 0.5 under the assumption that Theorem 0.6 and 0.7 hold.

Firstly, we define “quasi functors”.

Definition 2.1. *Let $\mathcal{C}_1, \mathcal{C}_2$ be categories. $F : \mathcal{C}_1 \rightarrow \mathcal{C}_2$ is a contravariant quasi functor if the following data are given.*

- (1) *For any $O \in \text{Obj}(\mathcal{C}_1)$, $F(O) \in \text{Obj}(\mathcal{C}_2)$.*
- (2) *For any $f \in \text{Mor}_{\mathcal{C}_1}(O_1, O_2)$, $F(f) \in \text{Mor}_{\mathcal{C}_2}(F(O_2), F(O_1))$ satisfying that $F(\text{id}) = \text{id}$.*

That is, a quasi functor is a “functor” which allows that $F(f_1 \circ f_2) \neq F(f_2) \circ F(f_1)$.

The classical category of Lie groups is too large to construct all quasi functors at the same time. For example, when $G = SO(3)$, *t.e.* $K(G, \tau)$ defined below does not sometimes have all information of G -equivariant twisted K -theory of G ([FHT4]).

2.1 Categories

Definition 2.2. *An object of $\mathcal{C}_0^{\text{twist}}$ and $T - \mathcal{C}^{\text{twist}}$ is a pair of a compact connected Lie group G with torsion-free π_1 and a positive central extension τ of LG . τ determines a G -equivariant twisting over G , for which we use the same character (Lemma 1.34).*

$f : (H, \tau') \rightarrow (G, \tau)$ is a morphism in $\mathcal{C}_0^{\text{twist}}$ if $G = H$, $\tau' = \tau$ and $f = \text{id}$. That is, $\mathcal{C}_0^{\text{twist}}$ is a discrete category.

$f : (H, \tau') \rightarrow (G, \tau)$ is a morphism in $T - \mathcal{C}^{\text{twist}}$ if it is a morphism in $\mathcal{C}_0^{\text{twist}}$ over H and G are tori, $\tau' = f^ \tau$, and f has the injective tangent map.*

We introduce the following definition to extend Theorem 0.6 to $f : H \rightarrow G$ satisfying the decomposable condition described below.

Let (G, τ) be an object of \mathcal{C}_0^{twist} and H be a compact connected Lie group with torsion-free π_1 . Take maximal tori S and T of H and G respectively. We can assume $f(S) \subseteq T$. That is, the following diagram commutes.

$$\begin{array}{ccc} H & \xrightarrow{f} & G \\ i \uparrow & & \uparrow k \\ S & \xrightarrow{f|_S} & T \end{array}$$

Definition 2.3. Let $f : H \rightarrow G$ be a smooth group homomorphism. The decomposable condition means the followings.

- (1) $f^*\tau$ is also positive.
- (2) $H/\ker(f)$ has torsion-free π_1 .
- (3) $[df(\mathfrak{h}), \mathfrak{s}^\perp] = 0$, where \mathfrak{s}^\perp is the orthogonal complement of $df(\mathfrak{s})$ in \mathfrak{g} under the bilinear form $\langle \cdot, \cdot \rangle_\tau$ defined in Lemma 1.10. We call this condition the “local condition”.

Remark 2.4. Condition (1) and (2) guarantee that $(H, f^*\tau)$ and $(H/\ker(f), f^*\tau)$ are objects of \mathcal{C}_0^{twist} .

Condition (3) guarantees that $H/\ker(f) \times S^\perp$ is also a group, where $S^\perp \subseteq T$ is the torus whose Lie algebra is \mathfrak{s}^\perp . Well-definedness of S^\perp is verified in Corollary 3.8.

If f satisfies this condition, df is automatically injective. It follows from Lemma 1.12.

The following examples satisfy the decomposable condition.

- Example 2.5.** (a) Any injection f when $\text{rank}(H) = \text{rank}(G)$.
(b) Any local injection f satisfying (2) when $\text{rank}(H) = \text{rank}(G)$.
(c) Any local injection f when H is a torus.
(d) $U(n) \hookrightarrow U(n+N)$ defined by

$$A \mapsto \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}.$$

Lemma 2.6. If f satisfies the local condition. The normalizer of S in H is mapped to the one of T in G .

Proof. Since H is compact and connected, $\exp : \mathfrak{h} \rightarrow H$ is surjective, that is, for any $h \in H$, there exists $v \in \mathfrak{h}$ such that $h = \exp(v)$. With the same way, for $t \in S^\perp$, there exists $\tilde{t} \in \mathfrak{s}^\perp$ such that $\exp(\tilde{t}) = t$. From Campbell-Hausdorff's formula

$$\begin{aligned} f(h)tf(h)^{-1} &= f(\exp(v))\exp(\tilde{t})f(\exp(-v)) \\ &= \exp(df(v) + \tilde{t} + \frac{1}{2}[df(v), \tilde{t}] + \cdots)f(\exp(-\tilde{t})) = \exp(df(v) + \tilde{t})\exp(-df(v)) \\ &= \exp(df(v) + \tilde{t} - df(v) + \frac{1}{2}[df(v) + \tilde{t}, -df(v)] + \cdots) \end{aligned}$$

$$= \exp(\widehat{t}) = t.$$

Therefore, $f(h)$ and t commute. Moreover, for any $x \in T$, there exist $s \in S$ and $t \in T^\perp$ such that $x = f(s) \cdot t$. We prove it at Proposition 3.9.

Therefore, for any $n \in N(S)$ and $x \in T$, the following holds.

$$\begin{aligned} f(n)xf(n^{-1}) &= f(n)(f(s)t)f(n^{-1}) = f(n)f(s)f(n^{-1})t \\ &= f(nsn^{-1})t \in T. \end{aligned}$$

□

From this lemma, we can define a homomorphism $f_* : W(H) \rightarrow W(G)$, where $W(H)$ and $W(G)$ are the Weyl groups of H and G respectively.

Lemma 2.7. *f_* is an injection.*

Proof. Suppose that $n \in N(S)$ and $[n] \in \ker(f_*)$. Then, $Ad(f(n))$ defines an automorphism of \mathfrak{t} . From the assumption that $[n] \in \ker(f_*)$, $Ad(f(n)) = id_{\mathfrak{t}}$. Since $Ad(f(n))|_{\mathfrak{s}} = Ad(n)$, $n \in S$. We obtain the conclusion. □

In this section, we define three quasi functors $t.e.K$, $char$ and $RL : \mathcal{C}_0^{twist} \rightarrow \mathcal{Ab}$. Later, we extend them as widely as possible.

Throughout this paper, \mathcal{Ab} is the category of abelian groups.

2.2 The quasi functor $char$

Let us start from the easiest quasi functor $char$. FHT isomorphism was verified by use of this quasi functor essentially.

Let us define a quasi functor

$$char : T - \mathcal{C}^{twist} \rightarrow \mathcal{Ab}.$$

Firstly, we define it from \mathcal{C}_0^{twist} , later we extend it to $T - \mathcal{C}^{twist}$.

Let (G, τ) be an object of \mathcal{C}_0^{twist} . Let T be a maximal torus of G . Let us recall that

$$1 \rightarrow U(1) \xrightarrow{i} T^\tau \xrightarrow{p} T \rightarrow 1$$

is the central extension of T induced from LT^τ . Λ_T^τ is the set of irreducible τ -twisted representations of T^τ . That is,

$$\begin{aligned} \Lambda_T^\tau &:= \{\lambda : T^\tau \rightarrow U(1) \mid \text{homomorphism such that } \lambda \circ i = id\} \\ &\cong \{d\lambda : \Pi_{T^\tau} \rightarrow \mathbb{Z} \mid \text{homomorphism such that } d\lambda \circ di = id\}. \end{aligned}$$

As we explain in Section 1, $W_{aff}^e(G) = \Pi_T \rtimes W(G)$ acts on Λ_T^τ , we can define regular orbits. An orbit is regular if the stabilizer is trivial.

Definition 2.8. *The quasi functor*

$$char : \mathcal{C}_0^{twist} \rightarrow \mathcal{Ab}$$

is defined by the \mathbb{Z} free module generated by the set of regular orbits $(\Lambda^\tau / W_{aff}^e(G))_{reg}$.

We write it as $\mathbb{Z}[(\Lambda^\tau / W_{aff}^e(G))_{reg}]$ also.

2.3 The quasi functor $t.e.K$

The following theorem is one of the main theorem of [FHT1]. We have already stated it for tori.

Theorem 2.9 ([FHT1]). *Let (G, τ) be an object of \mathcal{C}_0^{twist} .*

$$K_G^{\tau+k}(G) = \begin{cases} 0 & (k = \text{rank}(G) + 1 \pmod{2}) \\ \mathbb{Z}[(\Lambda^\tau / \kappa^\tau(W_{aff}^e(G)))_{reg}] & (k = \text{rank}(G) \pmod{2}) \end{cases}$$

This theorem implies the following immediately.

Corollary 2.10. *If $\text{rank}(G) - \text{rank}(H)$ is odd, the classical induced homomorphism $f^* \circ f^\natural : K_G^{\tau+k}(G) \rightarrow K_H^{f^*\tau+k}(H)$ is zero.*

Therefore, $f^* \circ f^\natural : K_G^{\tau+k}(G) \rightarrow K_H^{f^*\tau+k}(H)$ must not have a compatibility with the quasi functor $char$ generally.

Motivated by this observation, we define the quasi functor

$$t.e.K : T - \mathcal{C}^{twist} \rightarrow \mathcal{A}b$$

so that Theorem 0.6 holds.

Definition 2.11. *A quasi functor*

$$t.e.K : \mathcal{C}_0^{twist} \rightarrow \mathcal{A}b$$

is defined by $t.e.K(G, \tau) := K_G^{\tau+\text{rank}(G)}(G)$.

Theorem 2.9 implies that there is an isomorphism

$$M.d._G : t.e.K(G, \tau) \rightarrow char(G, \tau).$$

We define it in Section 4 by use of a Mackey decomposition and a push-forward map.

2.4 The quasi functor RL

We have already stated the following isomorphism (Theorem 1.19).

Theorem 2.12.

$$R^{\tau-\sigma}(LG) \cong \mathbb{Z}[(\Lambda^\tau / W_{aff}^e(G))_{reg}]$$

The isomorphism is given by the composition of taking the lowest weight and $\rho - \text{shift}$.

Let $f : H \rightarrow G$ be a smooth homomorphism, then we can define $Lf : LH \rightarrow LG$ by $Lf(l) := f \circ l$ for $l \in LH$.

At first sight, the homomorphism Lf seems to induce the pull back of representations $(Lf)^* : R^\tau(LG) \rightarrow R^{f^*\tau}(LH)$ just like the cases of compact Lie groups.

Recall that all irreducible representations of compact Lie groups are finite dimensional and all representations of compact Lie groups are completely reducible, therefore, we can identify finitely reducible representations as finite dimensional representations. It is independent of the groups that a representation is finite dimensional.

However, in the case of loop groups, “finitely reducibility” is not a property of spaces, but one of group actions unlike the cases of compact Lie groups. For example, if $i_1 : T \rightarrow T \times T'$ is the natural inclusion into the first factor, and (V, ρ) is an irreducible positive energy representation of $L(T \times T')$, $(V, (Li_1)^*\rho)$ is not finitely reducible (Corollary 5.6).

Motivated by this observation, we define a quasi functor

$$RL : T - \mathcal{C}^{twist} \rightarrow \mathcal{A}b.$$

Firstly, we define it at \mathcal{C}_0^{twist} , later we extend to $T - \mathcal{C}^{twist}$.

Definition 2.13. *The contravariant quasi functor $RL : \mathcal{C}_0^{twist} \rightarrow \mathcal{A}b$ is defined by*

$$RL(G, \tau) := R^{\tau-\sigma}(LG).$$

Let us recall that the isomorphism

$$l.w._G : RL(G, \tau) \rightarrow char(G, \tau)$$

has been defined by taking the lowest weight. We describe it more explicitly in Section 5 for tori.

We define the “induced homomorphism” $f^! = RL(f)$ for f which is a morphism in $T - \mathcal{C}^{twist}$ so that it has a compatibility with $char(f)$.

2.5 Rewriting of FHT isomorphism

Freed, Hopkins and Teleman constructed isomorphism between twisted equivariant K -theory and positive energy representation group by use of a family of “Dirac operators” parametrized by the set of connections over the trivial bundle $S^1 \times G$ in [FHT2]. We write FHT isomorphism for G as FHT_G .

We describe the essence of their proof for tori in our terminology.

Theorem 2.14 ([FHT2] Proposition 4.8). *Let T be a torus and τ be a positive central extension of LT . The following commutative diagram holds.*

$$\begin{array}{ccc} RL(T, \tau) & \xrightarrow{FHT_T} & t.e.K(T, \tau) \\ & \searrow l.w._T & \swarrow M.d._T \\ & char(T, \tau) & \end{array}$$

2.6 Proof of Theorem 0.5

Let us verify our main theorem under the assumption that Theorem 0.6 and 0.7 hold.

The following commutative diagram holds.

$$\begin{array}{ccccc}
 R^\tau(LT) & \xrightarrow{f^!} & R^{f^*\tau}(LT') & & \\
 \downarrow FHT_T & \searrow l.w._T & \swarrow l.w._{T'} & & \downarrow FHT_{T'} \\
 & (2) \quad char(T, \tau) & \xrightarrow{char(f)} & char(T', f^*\tau) & (3) \\
 & \nearrow M.d._T & \nwarrow M.d._{T'} & & \\
 K_T^{\tau+dim(T)}(T) & \xrightarrow{f^\#} & K_{T'}^{f^*\tau+dim(T')}(T') & &
 \end{array}$$

(1) is above the top arrow, (4) is below the bottom arrow.

Commutativity of (1) is Theorem 0.7, and commutativity of (4) is Theorem 0.6. Commutativity of (2) and (3) are the above theorem. Since $M.d._T$ and $M.d._{T'}$ are isomorphisms, we obtain the conclusion.

3 The quasi functor $char$

In this section, we study the easiest quasi functor $char$. We use Lemma 1.12 and 1.13.

3.1 For tori

In this section, we study in the case of tori. This is essential because we extend the quasi functor $char$ to general Lie groups by use of reduction to maximal tori.

3.1.1 Definition of induced homomorphism $char(f)$

Let $f : T' \rightarrow T$ be a local injection and τ be a positive central extension of LT . We define $char(f)([\lambda]_T)$ for $\lambda \in \Lambda_{\bar{T}}$, where $[\lambda]_T$ means the Π_T -orbit including λ . Let us consider the set $\{^t df(\lambda + \kappa^\tau(n)) | n \in \Pi_T\}$ and decompose it as $\Pi_{T'}$ -orbit. That is,

$$\{^t df(\lambda + \kappa^\tau(n)) | n \in \Pi_T\} = \coprod_{i=1}^N \{\mu_i + \kappa^{f^*\tau}(n) | n \in \Pi_{T'}\}.$$

N is a finite number because the set $\Lambda_{T'}^{f^*\tau} / \kappa^{f^*\tau}(\Pi_{T'})$ is a finite set.

Definition 3.1.

$$char(f)([\lambda]_T) := \sum_{i=1}^N [\mu_i]_{T'}$$

Lemma 3.2. *This is well-defined.*

Proof. It is sufficient to verify that $\{^t df(\lambda + \kappa^\tau(n)) | n \in \Pi_T\}$ is $\Pi_{T'}$ -invariant. It follows from that

$$^t df(\lambda + \kappa^\tau(n)) + \kappa^{f^* \tau}(l) = ^t df(\lambda + \kappa^\tau(n) + \kappa^\tau(df(l))) \in \{^t df(\lambda + \kappa^\tau(n)) | n \in \Pi_T\},$$

where the equality follows from Lemma 1.12. \square

We can compute the above homomorphism for some cases. We verify the followings in Section 5.

Lemma 3.3. *Let $i_1 : T_1 \rightarrow T_1 \times T_2$ be the natural inclusion into the first factor and $\tau = p_1^* \tau_1 + p_2^* \tau_2$, where τ_j is a positive central extension of LT_j and $p_j : T_1 \times T_2 \rightarrow T_j$ is the natural projection onto the j 'th factor. Then,*

$$\text{char}(i_1)([\lambda]_{T_1 \times T_2}) = [^t di_1(\lambda)]_{T_1}.$$

Lemma 3.4. *Let $q : T' \rightarrow T$ be a finite covering. Then,*

$$\text{char}(q)([\lambda]_T) = \sum_{m \in \Pi_T / df(\Pi_{T'})} [^t df(\lambda + \kappa^\tau(m))]_{T'}.$$

Remark 3.5. $m \in \Pi_T / df(\Pi_{T'})$ is a chosen representative element. Let us notice that the orbit $[^t df(\lambda + \kappa^\tau(m))]_{T'}$ is independent of the choice of the representative element.

3.1.2 A counterexample to that char is a functor

The quasi functor char is not a functor. We give a counterexample to that char is a functor.

Let us consider the following commutative diagram

$$\begin{array}{ccc} \mathbb{T} & \xrightarrow{h} & \mathbb{T}_2^2 \\ & \searrow f & \nearrow g \\ & \mathbb{T}_1^2 & \end{array}$$

where $f(z) := (z, z^{-1})$, $g(z_1, z_2) := (z_1 z_2, z_1 z_2^{-1})$ and $h(z) := (1, z^2)$. Let us notice that 1 is the unit element of \mathbb{T} .

We regard $\Pi_{\mathbb{T}} = \mathbb{Z} = \Lambda_{\mathbb{T}}^{h^* \tau}$, $\Pi_{\mathbb{T}_1^2} = \mathbb{Z}^2 = \Lambda_{\mathbb{T}_1^2}^{g^* \tau}$ and $\Pi_{\mathbb{T}_2^2} = \mathbb{Z}^2 = \Lambda_{\mathbb{T}_2^2}^{\tau}$ by fixing the base of Π and dual base.

Let us consider a central extension τ of $L\mathbb{T}_2^2$ so that

$$\kappa^\tau = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Therefore, $\text{char}(\mathbb{T}_2^2, \tau) = \mathbb{Z} \begin{bmatrix} 0 \\ 0 \end{bmatrix}_{\mathbb{T}_2^2} \cdot \begin{bmatrix} 0 \\ 0 \end{bmatrix}_{\mathbb{T}_2^2}$ means the orbit generated by $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$, that is, the whole $\Lambda_{\mathbb{T}_2^2}^\tau$.

We have the following data about the central extensions from $df = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$, $dg = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$, $dh = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$ and Lemma 1.12.

$$\kappa^{g^* \tau} = \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}, \kappa^{h^* \tau} = (-4).$$

Therefore,

$$\text{char}(\mathbb{T}_1^2, g^* \tau) = \mathbb{Z} \begin{bmatrix} 0 \\ 0 \end{bmatrix}_{\mathbb{T}_1^2} \oplus \mathbb{Z} \begin{bmatrix} 0 \\ 1 \end{bmatrix}_{\mathbb{T}_1^2} \oplus \mathbb{Z} \begin{bmatrix} 1 \\ 0 \end{bmatrix}_{\mathbb{T}_1^2} \oplus \mathbb{Z} \begin{bmatrix} 1 \\ 1 \end{bmatrix}_{\mathbb{T}_1^2}$$

$$\text{char}(\mathbb{T}, h^* \tau) = \mathbb{Z}[0]_{\mathbb{T}} \oplus \mathbb{Z}[1]_{\mathbb{T}} \oplus \mathbb{Z}[2]_{\mathbb{T}} \oplus \mathbb{Z}[3]_{\mathbb{T}}$$

Let us compute $\text{char}(f)$, $\text{char}(g)$ and $\text{char}(h)$. Since

$${}^t dh \begin{pmatrix} n \\ m \end{pmatrix} = 2m,$$

the image of $\begin{bmatrix} 0 \\ 0 \end{bmatrix}_{\mathbb{T}_2^2}$ under ${}^t dh$ is the set of the whole even numbers. Therefore,

$$\text{char}(h) \begin{bmatrix} 0 \\ 0 \end{bmatrix}_{\mathbb{T}_2^2} = [0]_{\mathbb{T}} + [2]_{\mathbb{T}}.$$

With the same way, we have the followings

$$\text{char}(g) \begin{bmatrix} 0 \\ 0 \end{bmatrix}_{\mathbb{T}_2^2} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}_{\mathbb{T}_1^2} + \begin{bmatrix} 1 \\ 1 \end{bmatrix}_{\mathbb{T}_1^2}$$

$$\text{char}(f) \begin{bmatrix} 0 \\ 0 \end{bmatrix}_{\mathbb{T}_1^2} = [0]_{\mathbb{T}} + [2]_{\mathbb{T}}$$

$$\text{char}(f) \begin{bmatrix} 1 \\ 1 \end{bmatrix}_{\mathbb{T}_1^2} = [0]_{\mathbb{T}} + [2]_{\mathbb{T}}.$$

Therefore,

$$\text{char}(f) \circ \text{char}(g) \begin{bmatrix} 0 \\ 0 \end{bmatrix}_{\mathbb{T}_2^2} = 2\text{char}(h) \begin{bmatrix} 0 \\ 0 \end{bmatrix}_{\mathbb{T}_2^2}.$$

3.1.3 The decomposition of a local injection associated with a positive central extension

While char is not a functor, it holds “functoriality” we need. To state it, we need to rewrite a local injection as a composition of finite coverings and an injection into the first factor of a direct product canonically.

Let T and T' be tori, $f : T' \rightarrow T$ be a local injection and τ be a positive central extension of LT . At first sight, our formulae Lemma 3.3, 3.4 are very special. However, we can rewrite any local injection f as a composition of finite coverings and an inclusion into the first factor of a direct product canonically. The following proposition is clear. Firstly, we state the following proposition, which is clear.

Proposition 3.6. *We can canonically decompose a local injection f as follows*

$$T' \xrightarrow{q} T' / \ker f \xrightarrow{i} T$$

where q is a finite covering, and i is an injection.

The above proposition tells us that we can assume that f is an injection.

Since we assume $f^*\tau$ is positive, $df(\mathfrak{t}') \cap df(\mathfrak{t}')^\perp = 0$, where $df(\mathfrak{t}')^\perp$ is the orthogonal completion under $\langle \cdot, \cdot \rangle_\tau$ in \mathfrak{t} .

Lemma 3.7. *$df(\mathfrak{t}')^\perp \cap \Pi_T$ is a \mathbb{Z} free module of rank $n - n'$, where $n = \dim(T)$ and $n' = \dim(T')$.*

Proof. Since Π_T is a free module, $df(\mathfrak{t}') \cap \Pi_T \subseteq \Pi_T$ is also free. Let us fix a \mathbb{Z} basis $\{v_1, v_2, \dots, v_{n'}\}$ of $\Pi_{T'}$. Then,

$$(df(\Pi_{T'}) \otimes \mathbb{Q})^\perp = (df(\mathfrak{t}')^\perp \cap \Pi_T) \otimes \mathbb{Q} = \bigcap_{i \in \{1, 2, \dots, n'\}} \ker(\kappa^\tau(v_i) \otimes \mathbb{Q}).$$

Since κ^τ is injective, $\dim((df(\Pi_{T'}) \otimes \mathbb{Q})^\perp) = n - n'$. So we can take a \mathbb{Q} basis $\{w_j\}_{j=1}^{n-n'}$ such that $w_j \in \Pi_T$. Since $\{w_j\}$ are linearly independent over \mathbb{Z} , $\text{rank}(df(\mathfrak{t}')^\perp \cap \Pi_T) \geq n - n'$. If $r := \text{rank}(df(\mathfrak{t}')^\perp \cap \Pi_T) > n - n'$, we can take a \mathbb{Z} basis $\{w'_j\}_{j=1}^r$ of $df(\mathfrak{t}')^\perp \cap \Pi_T$, linearly independent in $(df(\mathfrak{t}')^\perp \cap \Pi_T) \otimes \mathbb{Q}$ over \mathbb{Q} . The dimension of the right hand side is $n - n'$, which is a contradiction. \square

This lemma implies the followings.

Corollary 3.8.

$$T'^\perp := df(\mathfrak{t}')^\perp / (df(\mathfrak{t}')^\perp \cap \Pi_T) \subseteq T$$

is a torus. We call it the “orthogonal completion torus”. Let $j : T'^\perp \hookrightarrow T$ be the natural inclusion.

Proposition 3.9. *If $f : T' \rightarrow T$ is an injection, we can define the new torus $T' \times T'^\perp$ and decompose f as follows.*

$$T' \xrightarrow{i_1} T' \times T'^\perp \xrightarrow{f \cdot j} T.$$

Moreover, $f \cdot j$ is a finite covering, where $(f \cdot j)(t_1, t_2)$ is defined by $f(t_1) \cdot j(t_2)$.

Combining Proposition 3.6 with Proposition 3.9, we obtain the following theorem.

Theorem 3.10. *We can decompose f as follows*

$$T' \xrightarrow{q} T'/\ker(f) \xrightarrow{i_1} T'/\ker(f) \times (T'/\ker(f))^\perp \xrightarrow{f \cdot j} T,$$

where $(T'/\ker(f))^\perp$ is the orthogonal completion torus associated with f and τ , q is the natural finite covering, i_1 is the natural inclusion into the first factor and j is the natural inclusion.

Formula $\kappa^{(f \cdot j)^* \tau} = {}^t d(f \cdot j) \circ \kappa^\tau \circ d(f \cdot j)$ and that $df(\mathfrak{t}') \perp df(\mathfrak{t}'')^\perp$ under the bilinear form $\langle \cdot, \cdot \rangle_\tau$ imply the following lemma.

Lemma 3.11.

$$(f \cdot j)^* \tau = p_1^* i_1^* (f \cdot j)^* \tau + p_2^* i_2^* (f \cdot j)^* \tau = p_1^* \tau_1 + p_2^* \tau_2.$$

That is, the pair $((T'/\ker(f)) \times (T'/\ker(f))^\perp, (f \cdot j)^* \tau)$ satisfies the assumption of Lemma 3.3.

3.1.4 The functoriality we need

We use the same notations in the previous section.

The following is the functoriality we need mentioned above.

Theorem 3.12.

$$\text{char}(f) = \text{char}(q) \circ \text{char}(i_1) \circ \text{char}(f \cdot j)$$

Firstly, we verify that we can assume that f is injective.

Lemma 3.13. *If we have the following commutative diagram*

$$\begin{array}{ccc} T' & \xrightarrow{f} & T \\ & \searrow q & \nearrow g \\ & S & \end{array}$$

where f and g are local injections and q is a finite covering, the equality

$$\text{char}(f) = \text{char}(q) \circ \text{char}(g)$$

holds.

Proof. Let us take an orbit $[\lambda]_T \in \text{char}(T, \tau)$ and compute $\text{char}(f)([\lambda]_T)$ and $\text{char}(q) \circ \text{char}(g)([\lambda]_T)$.

Since ${}^t df = {}^t dq \circ {}^t dg$, the image of the orbit $[\lambda]_T$ under ${}^t df$ coincides with the one under ${}^t dq \circ {}^t dg$. Therefore, when $\text{char}(f)([\lambda]_T) = \sum_{i=1}^N [\mu_i]_{T'}$,

$$\text{char}(q) \circ \text{char}(g)([\lambda]_T) = \sum_{i=1}^N a_i [\mu_i]_{T'}$$

for some $a_i \in \mathbb{Z}_{>0}$. It is sufficient to verify that $a_i = 1$ for any i .

If $\text{char}(g)([\lambda]_T) = \sum_{j=1}^M [\nu_j]_S$,

$$\text{char}(q) \circ \text{char}(g)([\lambda]_T) = \sum_{j=1}^M \text{char}(q)([\nu_j]_S) = \sum_{i=1}^N a_i [\mu_i]_{T'},$$

where $a_i = \#\{j \in \{1, 2, \dots, M\} \mid {}^t dq(\nu_j + \kappa^{g^* \tau}(\Pi_S)) \supseteq \mu_i + \kappa^{f^* \tau}(\Pi_{T'})\}$.

However, if $[\nu]_S \neq [\nu']_S$, the images of orbits $\{\nu + \kappa^{g^* \tau}(n) \mid n \in \Pi_S\}$ and $\{\nu' + \kappa^{g^* \tau}(n) \mid n \in \Pi_S\}$ under ${}^t dq$ are disjoint because ${}^t dq$ is injective. That is, for any $j_1 \neq j_2$, $\text{char}(q)([\nu_{j_1}]_S) \neq \text{char}(q)([\nu_{j_2}]_S)$. \square

Let us verify Theorem 3.12. Let $f : T' \rightarrow T$ be an injection and τ be a positive central extension of LT .

Proof of Theorem 3.12. The following commutative diagram holds.

$$\begin{array}{ccc} T' & \xrightarrow{f} & T \\ & \searrow i_1 & \nearrow f \cdot j \\ & T' \times T'^{\perp} & \end{array}$$

Let $[\lambda]_T \in \text{char}(T, \tau)$ be an orbit, $\text{char}(f)([\lambda]_T) = \sum_{i=1}^N [\mu_i]_{T'}$, and $\text{char}(i_1) \circ \text{char}(f \cdot j)([\lambda]_T) = \sum_{i=1}^N a_i [\mu_i]_{T'}$. We verify that $a_i = 1$ for any i . Since

$$\begin{aligned} & {}^t d(f \cdot j)(\{\lambda + \kappa^{\tau}(n) \mid n \in \Pi_T\}) = \{ {}^t d(f \cdot j)(\lambda) + {}^t d(f \cdot j)(\kappa^{\tau}(n)) \mid n \in \Pi_T \} \\ = & \coprod_{m \in \Pi_T / d(f \cdot j)(\Pi_{T' \times T'^{\perp}})} \{ {}^t d(f \cdot j)(\lambda) + {}^t d(f \cdot j)(\kappa^{\tau}(m)) + \kappa^{(f \cdot j)^* \tau}(n') \mid n' \in \Pi_{T' \times T'^{\perp}} \} \\ = & \coprod_{m \in \Pi_T / d(f \cdot j)(\Pi_{T' \times T'^{\perp}})} [{}^t d(f \cdot j)(\lambda + \kappa^{\tau}(m))]_{T' \times T'^{\perp}}, \end{aligned}$$

that $a_i = 1$ for any i is equivalent to that

$$\text{char}(i_1)([{}^t d(f \cdot j)(\lambda + \kappa^{\tau}(m))]_{T' \times T'^{\perp}}) \neq \text{char}(i_1)([{}^t d(f \cdot j)(\lambda + \kappa^{\tau}(m'))]_{T' \times T'^{\perp}})$$

if $m \neq m'$. From linearity of these maps, it is sufficient to verify it under the assumption that $m' \in d(f \cdot j)(\Pi_{T' \times T'^{\perp}})$.

Let us assume that $[{}^t di_1({}^t d(f \cdot j)(\lambda + \kappa^{\tau}(m)))]_{T'} = [{}^t di_1({}^t d(f \cdot j)(\lambda))]_{T'}$. We verify that $m \in d(f \cdot j)(\Pi_{T' \times T'^{\perp}})$.

The assumption is equivalent to that

$${}^t di_1 \circ {}^t d(f \cdot j) \circ \kappa^{\tau}(m) = {}^t df(\kappa^{\tau}(m)) \in \kappa^{f^* \tau}(\Pi_{T'}).$$

Therefore, there exists $k \in \Pi_{T'}$ such that

$${}^t di_1 \circ {}^t d(f \cdot j) \circ \kappa^{\tau}(m) = {}^t di_1 \circ {}^t d(f \cdot j) \circ \kappa^{\tau} \circ df(k).$$

So, ${}^t d(f \cdot j)(\kappa^\tau(m) - \kappa^\tau(df(k))) \in \ker({}^t di_1)$.

Since $df(\mathfrak{t}') \perp df(\mathfrak{t}')^\perp$ under $\langle \cdot, \cdot \rangle_\tau$,

$$\ker({}^t di_1) = \kappa^{(f \cdot j)^* \tau}(di_1(\mathfrak{t}')^\perp).$$

Therefore, there exists $v \in di_1(\mathfrak{t}')^\perp = \mathfrak{t}'^\perp$ such that

$${}^t d(f \cdot j) \circ (\kappa^\tau(m) - \kappa^\tau(df(k))) = \kappa^{(f \cdot j)^* \tau}(v) = {}^t d(f \cdot j) \circ \kappa^\tau \circ d(f \cdot j)(v).$$

Since ${}^t d(f \cdot j) \circ \kappa^\tau$ is injective, $m - df(k) = d(f \cdot j)(v)$. Moreover, from that m and $df(k)$ are elements of Π_T , $m - df(k) = d(f \cdot j)(v) \in \Pi_{T'} \cap df(\mathfrak{t}')^\perp$.

By the definition of the orthogonal completion torus, $j = (f \cdot j)|_{T^\perp}$ is injective and

$$dj|_{\Pi_{T'^\perp}} = d(f \cdot j)|_{\Pi_{T'^\perp}} : \Pi_{T'^\perp} \rightarrow \Pi_T \cap df(\mathfrak{t}')^\perp$$

is bijective. Therefore, $v \in \Pi_{T'^\perp} \subseteq \Pi_{T' \times T'^\perp}$. Therefore,

$$m = d(f \cdot j)(di_1(k) + v) \in d(f \cdot j)(\Pi_{T' \times T'^\perp}).$$

□

3.2 For general Lie group

We extend the above construction to more general cases. We use reduction to maximal tori.

3.2.1 Reduction to a maximal torus

Let G be a compact connected Lie group with torsion-free π_1 and T be a maximal torus of G . Let $i : T \rightarrow G$ be the natural inclusion.

Let us suppose that $\lambda \in \Lambda_T^\tau$ determines a regular orbit in Λ_T^τ , that is, the stabilizer of λ in $W_{aff}^e(G)$ is trivial. Then the $W_{aff}^e(G)$ -orbit $W_{aff}^e(G) \cdot \lambda$ can be regarded as the union of Π_T -orbits $\coprod_{w \in W(G)} (w \cdot \lambda + \kappa^\tau(\Pi_T))$. So we define $char(i)([\lambda]_G)$ as follows.

Definition 3.14.

$$char(i)([\lambda]_G) := \sum_{w \in W(G)} [w \cdot \lambda]_T$$

The following lemma is clear from the definition.

Lemma 3.15. *char(i) is an injection.*

3.2.2 The definition of the induced homomorphism of char

Let us consider the following commutative diagram

$$\begin{array}{ccc} H & \xrightarrow{f} & G \\ i \uparrow & & \uparrow k \\ S & \xrightarrow{f|_S} & T \end{array}$$

where H and G are compact connected Lie groups with torsion-free π_1 , S and T are maximal tori of H and G respectively satisfying that $f(S) \subseteq T$. Let τ be a positive central extension of LG . Let us suppose that f satisfies the decomposable condition.

Definition 3.16. $\text{char}(f) : \text{char}(G, \tau) \rightarrow \text{char}(H, f^* \tau)$ is defined so that the following commutative diagram holds.

$$\begin{array}{ccc} \text{char}(G, \tau) & \xrightarrow{\text{char}(f)} & \text{char}(H, f^* \tau) \\ \text{char}(k) \downarrow & & \text{char}(i) \downarrow \\ \text{char}(T, k^* \tau) & \xrightarrow{\text{char}(f|_S)} & \text{char}(S, i^* f^* \tau). \end{array}$$

We have to verify well-definedness of this definition.

Firstly, we verify the regularity of orbits. For any $[\lambda]_G \in \text{char}(G, \tau)$, $\text{char}(f|_S) \circ \text{char}(k)([\lambda]_G)$ determines a subset in $\Lambda_S^{i^* f^* \tau}$. We verify that the subset is a union of $W_{\text{aff}}^e(H)$ -regular orbits. Since this subset coincides with the image of $W_{\text{aff}}^e(G) \cdot \lambda$ under ${}^t df$, it is sufficient to verify the following lemma.

Lemma 3.17. *If f satisfies the local condition, the image of a regular orbit under ${}^t df$ is $W_{\text{aff}}^e(H)$ -invariant and has the trivial stabilizer.*

Proof. Let $\lambda \in \Lambda_T^e$ determine a $W_{\text{aff}}^e(G)$ -regular orbit. Let us consider the $W_{\text{aff}}^e(G)$ -orbit $W_{\text{aff}}^e(G) \cdot \lambda$ and the image of it under ${}^t df$. Firstly, we verify that ${}^t df(W_{\text{aff}}^e(G) \cdot \lambda)$ is $W_{\text{aff}}^e(H)$ -invariant, then we verify that it has the trivial stabilizer.

For any $\mu \in {}^t df(W_{\text{aff}}^e(G) \cdot \lambda)$, there exists $\nu \in W_{\text{aff}}^e(G) \cdot \lambda$ such that $\mu = {}^t df(\nu)$. $W_{\text{aff}}^e(H)$ -invariance of ${}^t df(W_{\text{aff}}^e(G) \cdot \lambda)$ follows from that

$$w \cdot \mu = w \cdot {}^t df(\nu) = {}^t df(f_*(w) \cdot \nu) \in {}^t df(W_{\text{aff}}^e(G) \cdot \lambda)$$

$$\mu + \kappa^{f^* \tau}(n) = {}^t df(\nu + \kappa^\tau(df(n))) \in {}^t df(W_{\text{aff}}^e(G) \cdot \lambda)$$

where $w \in W(H)$ and $n \in \Pi_S$. These imply the invariance of ${}^t df(W_{\text{aff}}^e(G) \cdot \lambda)$.

Now, let us verify the triviality of the stabilizer of ${}^t df(W_{\text{aff}}^e(G) \cdot \lambda)$. Suppose that $(w, n) \in W_{\text{aff}}^e(H)$ stabilizes $\mu \in {}^t df(W_{\text{aff}}^e(G) \cdot \lambda)$. Then

$$\begin{aligned} w \cdot \mu + \kappa^{f^* \tau}(n) &= w \cdot {}^t df(\nu) + {}^t df \circ \kappa^\tau(df(n)) \\ &= {}^t df(f_*(w) \cdot \nu + \kappa^\tau(df(n))) = \mu = {}^t df(\nu). \end{aligned}$$

That is,

$$f_*(w) \cdot \nu + \kappa^\tau(df(n)) - \nu \in \ker({}^t df) = \kappa^\tau(df(\mathfrak{s})^\perp).$$

On the other hand, that

$$f_*(w) \cdot \nu + \kappa^\tau(df(n)) - \nu \in \kappa^\tau(df(\mathfrak{s}))$$

holds. It can be verified as follows.

Let us notice that ν can be orthogonally decomposed as $\nu = \nu_1 + \nu_2$, where $\nu_1 \in \kappa^\tau(df(\mathfrak{s}))$ and $\nu_2 \in \kappa^\tau(df(\mathfrak{s}))^\perp$. The local condition implies that $f_*(W(H))$ preserves $df(\mathfrak{s})$ and acts on \mathfrak{s}^\perp trivially. Therefore, $f_*(w).\nu - \nu = f_*(w).\nu_1 - \nu_1 \in \kappa^\tau(df(\mathfrak{s}))$.

Since $df(\mathfrak{s}) \cap df(\mathfrak{s})^\perp = 0$,

$$f_*(w).\nu + \kappa^\tau(df(n)) - \nu = 0.$$

Since ν has the trivial stabilizer, $(f_*(w), \kappa^\tau(df(n))) = (e_{W(G)}, 0)$. Since f_* , df and κ^τ are injective, we obtain the conclusion. \square

Let us verify the well-definedness of Definition 3.16 by the following three steps.

Lemma 3.18. *If the restriction of f to S is a finite covering, the above definition is well-defined.*

Proof. We can write $char(f|_S)$ explicitly as

$$char(f|_S)([\lambda]_T) = \sum_{m \in \Pi_T/df(\Pi_S)} [{}^t df(\lambda + \kappa^\tau(m))]_S.$$

Since ${}^t df$ is injective and the orbit $W_{aff}^e(G).\lambda$ is regular, for any $w, w' \in W(H)$ and $m, m' \in \Pi_T$ such that $(w, n) \neq (w', n')$, the following holds.

$$(w.{}^t df(\lambda + \kappa^\tau(m)) + \kappa^{i^* f^* \tau}(\Pi_S)) \cap (w'.{}^t df(\lambda + \kappa^\tau(m')) + \kappa^{i^* f^* \tau}(\Pi_S)) = \emptyset,$$

that is, $[{}^t df(w.\lambda + \kappa^\tau(m))]_S \neq [{}^t df(w'.\lambda + \kappa^\tau(m'))]_S$.

$$char(f|_S)(\sum_{w \in W(G)} [w.\lambda]_T) = \sum_{w \in W(G)} \sum_{m \in \Pi_T/df(\Pi_S)} [{}^t df(w.\lambda + \kappa^\tau(m))]_S.$$

Since Π_T is $W(G)$ -invariant and κ^τ is $W(G)$ -equivariant ($\langle \cdot, \cdot \rangle_\tau$ is $W(G)$ -invariant), for any $w \in W(G)$ and $m \in \Pi_T$, $w^{-1}.m \in \Pi_T$ and

$$w.\lambda + \kappa^\tau(m) = w.(\lambda + \kappa^\tau(w^{-1}.m)).$$

Let us consider the quotient space $W(G)/f_*(W(H)) = \{[W_1], [W_2], \dots, [W_k]\}$. That is, for any w , there uniquely exist $w' \in W(H)$ and $l \in \{1, 2, \dots, k\}$ such that $w = f_*(w')W_l$, where W_l is a fixed representative element of $[W_l]$. Since ${}^t df \circ f_*(w') = w'.{}^t df$,

$$\sum_{m \in \Pi_T/df(\Pi_S)} \sum_{w \in W(G)} [{}^t df(w.\lambda + \kappa^\tau(m))]_S = \sum_{l=1}^k \sum_{m \in \Pi_T/df(\Pi_S)} \sum_{w' \in W(H)} [w'.{}^t df(W_l.\lambda + \kappa^\tau(m))]_S.$$

By summing over $W(H)$, ambiguity of the choice of representatives W_1, W_2, \dots, W_k does not appear. $W(H)$ -equivariance of κ^τ implies that for any $m' \in \Pi_T/df(\Pi_S)$, there exists $m \in \Pi_T/df(\Pi_S)$ such that $w' \cdot \kappa^\tau(m) = \kappa^\tau(m')$. Since

$$\sum_{w' \in W(H)} [w' \cdot {}^t df(W_l \cdot \lambda + \kappa^\tau(m))]_S$$

is an element of $\text{char}(H, f^* \tau)$, so is the right hand side. \square

Lemma 3.19. *If $T = S \times S^\perp$, the restriction of f to S is the inclusion into the first factor and $k^* \tau = p_1^* i^* f^* \tau + p_2^* j^* k^* \tau$, where $p_1 : S \times S^\perp \rightarrow S$ and $p_2 : S \times S^\perp \rightarrow S^\perp$ are the natural projection, the above definition is well-defined.*

Proof. We can write $\text{char}(f|_S)$ explicitly as $\text{char}(f|_S)([\lambda]_T) = [{}^t df(\lambda)]_S$. So

$$\begin{aligned} \text{char}(f|_S) \left(\sum_{w \in W(G)} [w \cdot \lambda]_T \right) &= \sum_{w \in W(G)} [{}^t df(w \cdot \lambda)]_S \\ &= \sum_{[w] \in W(G)/W(H)} \sum_{w' \in W(H)} [w' \cdot {}^t df([w] \cdot \lambda)]_S \end{aligned}$$

Since $\sum_{w' \in W(H)} [w' \cdot {}^t df(W_l \cdot \lambda)]_S$ is an element of $\text{char}(H, f^* \tau)$, so is the right hand side. \square

Theorem 3.20. *The definition is well-defined for $f : H \rightarrow G$ satisfying the decomposable condition.*

Proof. Let us notice that we can decompose f so that the following commutative diagram holds.

$$\begin{array}{ccccccc} H & \xrightarrow{q} & H/\ker(f) & \xrightarrow{i_1} & H/\ker(f) \times S^\perp & \xrightarrow{\bar{f} \cdot j} & G \\ i \uparrow & & \bar{i} \uparrow & & \bar{i} \times id \uparrow & & k \uparrow \\ S & \xrightarrow{q|_S} & S/\ker(f|_S) & \xrightarrow{i_1|_S} & S/\ker(f|_S) \times S^\perp & \xrightarrow{\bar{f}|_S \cdot j} & T \end{array}$$

where $j : S^\perp \hookrightarrow T \subseteq G$ is the orthogonal completion torus defined in Corollary 3.8. Firstly, we assume that f is injective for simplicity.

Lemma 3.18 and 3.19 imply that the following commutative diagram holds.

$$\begin{array}{ccccc} \text{char}(G, \tau) & \xrightarrow{\text{char}(f \cdot j)} & \text{char}(H \times S^\perp, (f \cdot j)^* \tau) & \xrightarrow{\text{char}(i_1)} & \text{char}(H, f^* \tau) \\ \text{char}(k) \downarrow & & \text{char}(i \times id) \downarrow & & \text{char}(i) \downarrow \\ \text{char}(T, k^* \tau) & \xrightarrow{\text{char}(f|_S \cdot j)} & \text{char}(S \times S^\perp, (f|_S \cdot j)^* \tau) & \xrightarrow{\text{char}(i_1|_S)} & \text{char}(S, i^* f^* \tau) \end{array}$$

Theorem 3.12 tells us that $\text{char}(f|_S) = \text{char}(i_1|_S) \circ \text{char}(f|_S \cdot j)$.

If we define $\text{char}(f)$ by $\text{char}(i_1) \circ \text{char}(f \cdot j)$, $\text{char}(f)$ is compatible with $\text{char}(f|_S)$, that is, the following commutative diagram holds.

$$\begin{array}{ccc} \text{char}(G, \tau) & \xrightarrow{\text{char}(f)} & \text{char}(H, f^* \tau) \\ \text{char}(k) \downarrow & & \text{char}(i) \downarrow \\ \text{char}(T, k^* \tau) & \xrightarrow{\text{char}(f|_S)} & \text{char}(S, i^* f^* \tau) \end{array}$$

If f is not injective, we can verify the well-definedness with the same way. \square

3.2.3 An example which shows that we should not easily define $\text{char}(f)$ for general group

If one easily define $\text{char}(f)$, just like the case in tori, $\text{char}(f)([\lambda]_G)$ may be the sum of $W_{\text{aff}}^e(H)$ -orbits ${}^t df(W(G).(\lambda + \kappa^\tau(\Pi_T)))$. However, this is not compatible with the case of tori. We give an example.

Let us define $f : \mathbb{T} \rightarrow U(3)$ as

$$f(z) := \begin{pmatrix} z & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and τ be a positive central extension of $LU(3)$ such that induced homomorphism is represented as

$$\kappa^\tau = \begin{pmatrix} -3 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -3 \end{pmatrix}$$

where the maximal torus of $U(3)$ is the diagonal matrices. Let $i : \mathbb{T}^3 \rightarrow U(3)$ be the natural inclusion. We write f as i_1 if we regard f as $\mathbb{T} \rightarrow \mathbb{T}^3$. That is, the following commutative diagram holds.

$$\begin{array}{ccc} \mathbb{T} & \xrightarrow{f} & U(3) \\ \parallel & & \uparrow i \\ \mathbb{T} & \xrightarrow{i_1} & \mathbb{T}^3 \end{array}$$

Since $W(U(3)) \cong \mathfrak{S}_3$, $W_{\text{aff}}^e(U(3))$ -regular orbit is $W_{\text{aff}}^e(U(3)) \cdot \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$. Therefore, $K_{U(3)}^{\tau+3}(U(3)) \cong \mathbb{Z}$.

Since $\kappa^{f^* \tau} = (-3)$, $K_{\mathbb{T}}^{f^* \tau+1}(\mathbb{T}) \cong \mathbb{Z}^3$.

We can compute $\text{char}(i)$ and $\text{char}(i_1)$ as follows.

$$\text{char}(i) \left(\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}_{U(3)} \right) = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}_{\mathbb{T}^3} + \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}_{\mathbb{T}^3} + \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}_{\mathbb{T}^3} + \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}_{\mathbb{T}^3} + \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}_{\mathbb{T}^3} + \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}_{\mathbb{T}^3}.$$

$$\text{char}(i_1)\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix}_{\mathbb{T}^3}\right) = [a]_{\mathbb{T}}$$

$$\text{Therefore, } \text{char}(i_1) \circ \text{char}(i)\left(\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}_{U(3)}\right) = 2([0]_{\mathbb{T}} + [1]_{\mathbb{T}} + [2]_{\mathbb{T}}).$$

$$\text{However, } {}^t df(W_{\text{aff}}^e(U(3)) \cdot \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}) \text{ is entire of } \Lambda_{\mathbb{T}}^{f^* \tau}, \text{ that is } \text{char}(f)\left(\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}_{U(3)}\right)$$

seems to be $[0]_{\mathbb{T}} + [1]_{\mathbb{T}} + [2]_{\mathbb{T}}$. This example tells us that we should not define just like the case of tori using the correspondence of orbits.

4 The quasi functor $t.e.K$

4.1 General constructions

In this section, we describe general constructions, Künneth isomorphisms, family index maps and Mackey decompositions.

Künneth isomorphisms will be verified by a canonical way, that is, Mayer Vietoris argument just like ordinary K -theory [Karoubi]. By use of it, we define family index maps.

A Mackey decomposition is an analogue of the isomorphism in un-twisted case $K_G(X) \cong K(X) \otimes R(G)$ for trivial G space X .

4.1.1 Künneth isomorphisms and family index maps

In this section, we verify Künneth isomorphisms and define family index maps for twisted equivariant K -theory in special cases. This is very important to construct $i_1^\#$, where i_1 is the natural inclusion into the first factor of the direct product.

Let us start from more general construction, the tensor product.

Definition 4.1. *We can define the graded commutative and associative tensor product*

$$K_G^{\tau_1+n_1}(X) \otimes K_G^{\tau_2+n_2}(X) \rightarrow K_G^{\tau_1+\tau_2+n_1+n_2}(X)$$

by well known formula

$$[\{\mathcal{F}_x\}_{x \in X}] \otimes [\{\mathcal{G}_x\}_{x \in X}] \mapsto [\{\mathcal{F}_x \otimes id + \epsilon \otimes \mathcal{G}_x\}_{x \in X}],$$

where ϵ is the grading involution $id \oplus (-id)$. Associativity follows from Kuiper's theorem.

Remark 4.2. (1) *When G acts on X trivially, we have a natural map $K^{\tau_2+n_2}(X) \rightarrow K_G^{\tau_2+n_2}(X)$ by regarding a family of Fredholm operators equivariant one via the trivial action. Through this map, we can define the tensor product*

$$K_G^{\tau_1+n_1}(X) \otimes K^{\tau_2+n_2}(X) \rightarrow K_G^{\tau_1+\tau_2+n_1+n_2}(X).$$

We need it for Künneth isomorphism in our cases.

(2) We can define a “formal K -theory class” $\{\mathcal{F}_x\}_{x \in X}$ as a family of Fredholm operators not compactly supported. Moreover, we can define as

$$[\{\mathcal{G}_x\}_{x \in X}] \otimes \{\mathcal{F}_x\}_{x \in X} := [\{\mathcal{G}_x \otimes id + \epsilon \otimes \mathcal{F}_x\}_{x \in X}]$$

for a K -theory class $[\{\mathcal{G}_x\}_{x \in X}] \in K_G^{\tau+n}(X)$ just like de-Rham complex $\Omega^n(X) \otimes \Omega_c^m(X) \rightarrow \Omega_c^{n+m}(X)$. We need this construction for a Mackey decomposition in Section 4.1.2.

The following is clear from the definition.

Proposition 4.3. F^* and f^\natural preserves \otimes .

Let us verify Künneth isomorphisms.

Theorem 4.4. Let M be a smooth compact manifold which G acts on trivially, τ be a G -equivariant twisting over M and Z be an n dimensional compact manifold. Let us suppose that $K^m(Z)$ is free abelian groups for any m . Let us consider the direct product $M \times Z$ and the projections $p_1 : M \times Z \rightarrow M$, $p_2 : M \times Z \rightarrow Z$. G acts on $M \times Z$ trivially. Then the homomorphism

$$p_1^* \otimes p_2^* : K_G^{\tau+*}(M) \otimes K^*(Z) \rightarrow K_G^{p_1^* \tau+*}(M \times Z)$$

is an isomorphism. Therefore, we can define the Künneth isomorphism

$$\Phi_{M \times Z, G, \tau} : K_G^{p_1^* \tau+*}(M \times Z) \xrightarrow{\cong} K_G^{\tau+*}(M) \otimes K^*(Z)$$

by $(p_1^* \otimes p_2^*)^{-1}$, where K^* means $K^0 \oplus K^1$.

Proof. We write the restriction of a twisting τ to U which is a subset of M as $\tau|_U$ for simplicity.

We can take a finite compact contractible cover $\{U_i\}_{i=1}^N$ of M such that $\bigcap_{i \in I} U_i$ is contractible for any $I \subseteq \{1, 2, \dots, N\}$, for example triangulation.

Since U_i is compact and contractible, we can assume that U_i is a point $\{pt\}$ and τ determines the central extension G^τ of G . Firstly, we verify that

$$p_1^* \otimes p_2^* : K_G^{\tau|_{\{pt\}}+*}(\{pt\}) \otimes K^*(Z) \xrightarrow{\cong} K_G^{p_1^* (\tau|_{\{pt\}})+*}(\{pt\} \times Z)$$

is an isomorphism.

Since Z is a compact manifold, it has a finite compact contractible cover $\{W_j\}_{j=1}^{N'}$ such that $\bigcap_{j \in J} W_j$ is contractible for any $J \subseteq \{1, 2, \dots, N'\}$, for example triangulation. Since W_j is compact and contractible, we can assume that W_j is also a point.

$$\begin{aligned} K_G^{\tau|_{\{pt\}}+*}(\{pt\}) \otimes K^*(\{pt\}) &\cong K_{G^\tau}^*(\{pt\})(1) \otimes K^*(\{pt\}) \\ &\cong K_{G^\tau}^*(\{pt\} \times \{pt\})(1) \cong K_G^{p_1^* (\tau|_{\{pt\}})+*}(\{pt\} \times \{pt\}), \end{aligned}$$

where (1) means that the weight of $i(U(1)) \subseteq G^\tau$ is 1, that is, $K_{G^\tau}^*(\{pt\})(1)$ is the Grothendieck completion of the semigroup of G^τ -equivariant vector spaces which $i(U(1)) \subseteq G^\tau$ acts on by scalar multiplication. The addition of this semigroup is defined by the direct sum. The first and the last isomorphisms are verified in [FHT1] Proposition 3.5 i). The middle isomorphism is obtained by $p_1^* \otimes p_2^*$.

These isomorphisms and Mayer Vietoris argument imply that

$$p_1^* \otimes p_2^* : K_G^{\tau|_{\{pt\}}+*}(\{pt\}) \otimes K^*(W_1 \cup W_2) \xrightarrow{\cong} K_G^{p_1^*(\tau|_{\{pt\}})+*}(\{pt\} \times (W_1 \cup W_2))$$

is an isomorphism. By the induction on the number of k in $\bigcup_{i=1}^k W_j$, we have the conclusion.

With the same way, we can verify that

$$p_1^* \otimes p_2^* : K_G^{\tau|_{U_1 \cup U_2}+*}(U_1 \cup U_2) \otimes K^*(Z) \xrightarrow{\cong} K_G^{p_1^*(\tau|_{U_1 \cup U_2})+*}((U_1 \cup U_2) \times Z)$$

is an isomorphism. By the induction on the number of k in $\bigcup_{i=1}^k U_i$, we have the conclusion. \square

Let us define family index maps for our cases.

When the tangent bundle TZ is trivial, and we fix a trivialization of TZ and an orientation of Z , we can define isomorphisms

$$K^n(Z) \cong K^{TZ}(Z) \cong K(Z, Cl(TZ))$$

by fixing the trivial Spinor bundle $Z \times \Delta_{\mathbb{R}^n}$, where n is the dimension of Z (Example 1.29). Let us recall that an orientation of vector space V determines the Spinor Δ_V ([Furuta] Theorem 2.15).

The following lemma is one of the consequence of Atiyah-Singer index theorem when n is even.

Lemma 4.5 ([CW]). *We can define the index map $ind_Z : K^n(Z) \rightarrow \mathbb{Z}$. It depends on the choice of the isomorphism $K^n(Z) \cong K(Z, Cl(TZ))$, that is, it depends on the choice of the trivialization of TZ and the orientation of Z . ind_Z is given by the push-forward along the unique map $Z \rightarrow \{pt\}$.*

For simplicity, we assume that $K_G^\tau(M)$ or $K_G^{\tau+1}(M)$ is equal to 0. Then we can define a family index map. We take $n_0 = 0$ or 1 so that $K_G^{\tau+n_0+1}(M) = 0$.

Definition 4.6. *If we fix a trivialization of TZ and an orientation of Z ,*

$$ind_{M \times Z \rightarrow M} : K_G^{p_1^*\tau+n_0+n}(M \times Z) \rightarrow K_G^{\tau+n_0}(M)$$

is defined by the composition of the following sequence

$$K_G^{p_1^*\tau+n_0+n}(M \times Z) \xrightarrow{\Phi_{M \times Z, G, \tau}} K_G^{\tau+n_0}(M) \otimes K^n(Z) \xrightarrow{id \otimes ind_Z} K_G^{\tau+n_0}(M).$$

We can extend these constructions to non-trivial G -manifold M .

Theorem 4.7. *Theorem 4.4 is valid for non-trivial G -manifold M .*

Proof. We have the “slice theorem”. That is, each $x \in M$ has a closed G -neighborhood of the form $G \times_{G_x} S_x$, where the slice S_x is equivariantly contractible under the stabilizer G_x . Therefore, we have an isomorphism

$$\begin{aligned} K_G^{p_1^*(\tau|_{G \times_{G_x} S_x})+*}((G \times_{G_x} S_x) \times Z) &\cong K_G^{p_1^*(\tau|_{G \times_{G_x} S_x})+*}(G \times_{G_x} (S_x \times Z)) \\ &\cong K_{G_x}^{p_1^*(\tau|_{S_x})+*}(S_x \times Z), \end{aligned}$$

where the last isomorphism can be defined by the pull back along the natural inclusion

$$S_x \times Z \ni (s, z) \mapsto [(e_G, s, z)] \in G \times_{G_x} (S_x \times Z).$$

Since S_x is equivariantly contractible under the stabilizer G_x , we have an isomorphism

$$K_{G_x}^{p_1^*(\tau|_{S_x})+*}(S_x \times Z) \cong K_{G_x}^{p_1^*(\tau|_{\{x\}})+*}(\{x\} \times Z).$$

$K_{G_x}^{p_1^*(\tau|_{\{x\}})+*}(\{x\} \times Z)$ is isomorphic to $K_{G_x}^{\tau|_{\{x\}}+*}(\{x\}) \otimes K^*(Z)$ by Theorem 4.4.

From the isomorphism $K_{G_x}^{\tau|_{\{pt\}}+*}(\{x\}) \cong K_G^{\tau|_{G \times_{G_x} S_x}+*}(G \times_{G_x} S_x)$, the natural isomorphism

$$K_G^{p_1^*(\tau|_{G \times_{G_x} S_x})+*}(G \times_{G_x} (S_x \times Z)) \cong K_G^{\tau|_{G \times_{G_x} S_x}+*}(G \times_{G_x} S_x) \otimes K^*(Z)$$

holds. Mayer Vietoris argument implies the theorem just like Theorem 4.4. \square

When $K_G^{\tau+n_0+1}(M) = 0$ and Z has the trivial tangent bundle, we can extend a family index map to non-trivial G -manifold with the same way in the case of trivial action.

Definition 4.8. *A family index map*

$$ind_{M \times Z \rightarrow M} : K_G^{p_1^*\tau+n_0+n}(M \times Z) \rightarrow K_G^{p_1^*\tau+n_0}(M)$$

is defined just like Definition 4.6.

The following lemmas about functorialities of Künneth isomorphisms are clear from Proposition 4.3.

Let X and Y be smooth manifolds, G be a compact Lie group and G act on X and Y smoothly. Let a smooth proper map $F : Y \rightarrow X$ be G -equivariant, $f : H \rightarrow G$ be a smooth group homomorphism and τ be a G -equivariant twisting over X , which is also H -equivariant through f . When we regard τ as H -equivariant one, we write it as $f^{\natural}\tau$.

Lemma 4.9. *The following commutative diagram holds.*

$$\begin{array}{ccc}
K_G^{p_1^* \tau + *}(M \times Z) & \xrightarrow{\Phi_{M \times Z, G, \tau}} & K_G^{\tau + *}(M) \otimes K^*(Z) \\
(F \times id)^* \downarrow & & F^* \otimes id \downarrow \\
K_G^{p_1^* F^* \tau + *}(N \times Z) & \xrightarrow{\Phi_{N \times Z, G, F^* \tau}} & K_G^{F^* \tau + *}(N) \otimes K^*(Z)
\end{array}$$

Lemma 4.10. *The following commutative diagram holds.*

$$\begin{array}{ccc}
K_G^{p_1^* \tau + *}(M \times Z) & \xrightarrow{\Phi_{M \times Z, G, \tau}} & K_G^{\tau + *}(M) \otimes K^*(Z) \\
f^\natural \downarrow & & f^\natural \otimes id \downarrow \\
K_H^{p_1^* f^\natural \tau + *}(M \times Z) & \xrightarrow{\Phi_{M \times Z, H, f^\natural \tau}} & K_H^{f^\natural \tau + *}(M) \otimes K^*(Z)
\end{array}$$

These lemmas imply the followings.

Proposition 4.11. *The following commutative diagram holds.*

$$\begin{array}{ccc}
K_G^{p_1^* \tau + n_0 + n}(M \times Z) & \xrightarrow{ind_{M \times Z \rightarrow M}} & K_G^{\tau + n_0}(M) \\
(F \times id)^* \downarrow & & F^* \downarrow \\
K_G^{p_1^* F^* \tau + n_0 + n}(N \times Z) & \xrightarrow{ind_{N \times Z \rightarrow N}} & K_G^{F^* \tau + n_0}(N)
\end{array}$$

Proof. Lemma 4.9 implies the commutativity of the left side of the following diagram.

$$\begin{array}{ccccc}
K_G^{p_1^* \tau + n_0 + n}(M \times Z) & \xrightarrow{\Phi_{M \times Z, G, \tau}} & K_G^{\tau + n_0}(M) \otimes K^n(Z) & \xrightarrow{id \otimes ind_Z} & K_G^{\tau + n_0}(M) \\
(F \times id)^* \downarrow & & F^* \otimes id \downarrow & & F^* \downarrow \\
K_G^{p_1^* F^* \tau + n_0 + n}(N \times Z) & \xrightarrow{\Phi_{N \times Z, G, \tau}} & K_G^{F^* \tau + n_0}(N) \otimes K^n(Z) & \xrightarrow{id \otimes ind_Z} & K_G^{F^* \tau + n_0}(N)
\end{array}$$

The compositions of horizontal arrows are $ind_{M \times Z \rightarrow M}$ and $ind_{N \times Z \rightarrow N}$ respectively. The commutativity of the right side is clear. \square

Proposition 4.12. *The following commutative diagram holds.*

$$\begin{array}{ccc}
K_G^{p_1^* \tau + n_0 + n}(M \times Z) & \xrightarrow{ind_{M \times Z \rightarrow M}} & K_G^{\tau + n_0}(M) \\
f^\natural \downarrow & & f^\natural \downarrow \\
K_H^{p_1^* f^\natural \tau + n_0 + n}(M \times Z) & \xrightarrow{ind_{M \times Z \rightarrow M}} & K_H^{f^\natural \tau + n_0}(M)
\end{array}$$

Proof. We can verify it with the same way in the proof of the above proposition from Lemma 4.10. \square

4.1.2 A Mackey decomposition

We use a Mackey decomposition to compute $K_T^{\tau+\dim(T)}(T)$ for a torus, which is an analogue of the isomorphism in un-twisted equivariant K -theory $K_G(X) \cong K(X) \otimes R(G)$ for trivial G -space X . We state it for trivial T -spaces. We omit the proof. See [FHT3] for detail.

Let T be a torus, X be a smooth manifold, T act on X trivially, P be a T -equivariant projective bundle over X , and τ be the twisting represented by P . Covering space Y and the associated twisting τ' over Y are defined by the following.

- (i) a T -equivariant family, parametrized by X , of central extension T^τ of T by $U(1)$;
- (ii) a covering space $p : Y \rightarrow X$, whose fibers label the isomorphism classes of irreducible, τ -twisted representations Λ_T^τ ;
- (iii) a tautological projective bundle $\mathbb{P}R \rightarrow Y$ whose fiber $\mathbb{P}R_\lambda$ at $\lambda \in Y$ is the projectification of the τ -twisted representation of T labelled by λ ;
- (iv) a formal $\mathbb{P}R$ -twisted K -theory class $[R]$, represented by R ;
- (v) a non-equivariant twisting τ' over Y and an isomorphism as T -equivariant twistings $\tau' \cong p^*\tau - \mathbb{P}R$.

Theorem 4.13 ([FHT3]). *There is an isomorphism*

$$Mackey_{X,T,\tau} : K_T^{\tau+k}(X) \rightarrow K^{\tau'+k}(Y)$$

associated with an isomorphism $\tau' \cong p^\tau - \mathbb{P}R$ in (v).*

We verify naturality with respect to pull back along a continuous map between two spaces and pull back of group action along a local injection between two tori. Before that, we verify Theorem 2.9.

Corollary 4.14. *Let T be a torus and τ be a positive central extension of LT and the associated T -equivariant twisting over T . Fix an orientation of T .*

Then we have an isomorphism $M.d.T : K_T^{\tau+\dim(T)}(T) \rightarrow \text{char}(T, \tau)$. Moreover, $K_T^{\tau+\dim(T)+1}(T) \cong 0$.

Proof. Associated covering space Y is $\Lambda_T^\tau \times_{\Pi_T} \mathfrak{t}$, Π_T acts on Λ_T^τ through κ^τ (Definition 1.9) and on \mathfrak{t} via the translation. $\kappa^\tau(n)$ represents how the character changes when one travels in T along the geodesic loop n . Since τ is positive, κ^τ is injective. Therefore, $\Lambda_T^\tau \times_{\Pi_T} \mathfrak{t}$ has a structure of a trivial vector bundle over $\Lambda_T^\tau / \kappa^\tau(\Pi_T)$ whose fiber is \mathfrak{t} . We write this vector bundle as

$$\pi : \Lambda_T^\tau \times_{\Pi_T} \mathfrak{t} \rightarrow \Lambda_T^\tau / \kappa^\tau(\Pi_T).$$

Moreover, since any connected component of $\Lambda_T^\tau \times_{\Pi_T} \mathfrak{t}$ is contractible, a twisting over $\Lambda_T^\tau \times_{\Pi_T} \mathfrak{t}$ is automatically trivial. So we have the following series of isomorphisms

$$K_T^{\tau+\dim(T)}(T) \xrightarrow{Mackey_{Y,T,\tau}} K^{\dim(T)}(\Lambda_T^\tau \times_{\Pi_T} \mathfrak{t}) \xrightarrow{\pi!} K(\Lambda_T^\tau / \kappa^\tau(\Pi_T)) \cong \text{char}(T, \tau).$$

The first isomorphism is the Mackey decomposition and $\pi_!$ is Thom isomorphism, so this isomorphism depends on the orientation of T .

Since $K^1(\Lambda_T^\tau / \kappa^\tau(\Pi_T)) \cong 0$, $K_T^{\tau + \dim(T) + 1}(T) \cong 0$. \square

We omit the proof of the following theorem. You can find the proof in [FHT3].

Theorem 4.15. *Let $k : T \hookrightarrow G$ be a chosen maximal torus. We can define the isomorphism $M.d.G : K_G^{\tau + \text{rank}(G)}(G) \rightarrow \text{char}(G, \tau)$ so that the following diagram commutes.*

$$\begin{array}{ccc} t.e.K(G, \tau) & \xrightarrow{k^* \circ k^\natural} & t.e.K(T, k^* \tau) \\ M.d.G \downarrow & & M.d.T \downarrow \\ \text{char}(G, \tau) & \xrightarrow{\text{char}(k)} & \text{char}(T, k^* \tau) \end{array}$$

Let us construct a Mackey decomposition, and verify naturality with respect to F^* and f^\natural , where F is a proper smooth map between two manifolds and f is a locally injective group homomorphism.

Construction: The isomorphism stated above is the inverse of the composition of the following sequence

$$K^{\tau'}(Y) \rightarrow K_T^{\tau'}(Y) \cong K_T^{p^* \tau - \mathbb{P}R}(Y) \xrightarrow{\otimes [R]} K_T^{p^* \tau}(Y) \xrightarrow{p_!} K_T^\tau(X),$$

where the first map is defined by regarding a family of Fredholm operators as an equivariant one via the trivial action, the second isomorphism is defined by the isomorphism of twistings, the third map is defined by the tensor product with the formal K -theory class $[R]$, and the last map is the push-forward along $p : Y \rightarrow X$.

Since the central extension of T splits and T is abelian group, T^τ is also an abelian group ([FHT1] Lemma 4.1). So τ -twisted irreducible representations of T are 1 dimensional. Therefore, the fiber $\mathbb{P}R_\lambda$ of $\mathbb{P}R$ at λ is a point, that is, $\mathbb{P}R = Y$. In other words, the twisting $\mathbb{P}R \cong 0$.

Remark 4.16. *If T -equivariant twistings τ and σ are isomorphic, the isomorphism $K_T^\tau(X) \rightarrow K_T^\sigma(X)$ depends on the choice of the “homotopy class” of isomorphisms between τ and σ (Proposition 1.33). “Homotopy classes” of isomorphisms are classified by the second equivariant cohomology group $H_T^2(X)$, that is, isomorphism classes of equivariant complex line bundles, just as $H^2(X)$ acts on $K^*(X)$ via the tensor product with the corresponding complex line bundle, where we regard $K^*(X)$ as not a ring but a group.*

So the isomorphism $K_T^{\tau'}(Y) \cong K_T^{p^ \tau - \mathbb{P}R}(Y)$ is determined not automatically but by taking an isomorphism $\tau' \cong p^* \tau - \mathbb{P}R$.*

Let us verify naturality of a Mackey decomposition.

Theorem 4.17. *Let X and X' be manifolds, T be a torus acting on X and X' trivially, τ be a T -equivariant twisting over X , and $F : X' \rightarrow X$ be a smooth proper map. $p : Y_T \rightarrow X$ and $p' : Y'_T \rightarrow X'$ are the covering spaces constructed above.*

The following commutative diagram holds.

$$\begin{array}{ccc} K^{\tau'+k}(Y_T) & \xrightarrow{\tilde{F}^*} & K^{\tilde{F}^*\tau'+k}(Y'_T) \\ \text{Mackey}_{X,T,\tau} \uparrow & & \uparrow \text{Mackey}_{X',T,F^*\tau} \\ K_T^{\tau+k}(X) & \xrightarrow{F^*} & K_T^{F^*\tau+k}(X') \end{array}$$

where \tilde{F} is defined in the proof.

Proof. Firstly, we verify naturality of the covering space.

Lemma 4.18. *The covering space Y'_T over X' associated with $F^*\tau$ is homeomorphic to F^*Y_T . Therefore, we can define a continuous map $\tilde{F} : Y'_T \rightarrow Y_T$ and the following commutative diagram holds.*

$$\begin{array}{ccc} Y'_T \cong F^*Y_T & \xrightarrow{\tilde{F}} & Y_T \\ p' \downarrow & & \downarrow p \\ X' & \xrightarrow{F} & X \end{array}$$

Proof. Firstly, we verify naturality of T^τ family with F^* , then we verify the lemma.

Let P be a projective bundle which represents the twisting τ and ρ be the action of T on P . T_x^τ is defined by the pull back of the central extension $U(1) \rightarrow U(\mathcal{H}_x) \rightarrow PU(\mathcal{H}_x)$ along the homomorphism $\rho_x : T \rightarrow U(P_x) = PU(\mathcal{H}_x)$, that is, $T_x^\tau = \rho_x^*(U(\mathcal{H}_x))$, where \mathcal{H}_x is a chosen lift of P_x .

Since the action of T on F^*P is defined by $\rho_{x'} = \rho_{F(x')}$, $T_{x'}^{F^*\tau} = \rho_{x'}^*(U(\mathcal{H}_{F(x')})) = \rho_{F(x')}^*(U(\mathcal{H}_{F(x')})) = T_{F(x')}^\tau$.

Let us verify the lemma. Let us recall the topology of Y_T . Let \mathcal{U} be a sufficient small open neighborhood of $x_0 \in X$ in X . Then we can trivialize P on \mathcal{U} , that is, $P|_{\mathcal{U}} \cong \mathcal{U} \times \mathbb{P}(\mathcal{H})$. By use of this homeomorphism, we can trivialize the T^τ family over \mathcal{U} , that is, $\{T_x^\tau\}_{x \in \mathcal{U}} \cong \mathcal{U} \times T_{x_0}^\tau$. So we can trivialize as $Y_T|_{\mathcal{U}} \cong \mathcal{U} \times \Lambda_{T,x_0}^\tau$. Since this trivialization depends only on one of P , we can define the associated trivialization of $Y_{T'}$ on $F^{-1}(\mathcal{U})$ by well known method.

Since $\Lambda_{T,F(x')}^\tau = \Lambda_{T,x'}^{F^*\tau}$, we can define a fiber map $\tilde{F} : Y'_T \rightarrow Y_T$ by $\tilde{F}(x', \lambda) := (F(x'), \lambda)$, where $x' \in X'$, $\lambda \in \Lambda_{T,x'}^{F^*\tau}$. Let us notice that $p \circ \tilde{F} = F \circ p'$ from the definition of \tilde{F} . Continuity of this map follows from that the restriction of \tilde{F} to $F^{-1}(\mathcal{U})$ can be written as $F|_{F^{-1}(\mathcal{U})} \times id_{\Lambda_{T,x_0}^\tau}$ through the above trivialization. By the universality of the fiber product, there exists the unique continuous fiber map $F^*Y_T \rightarrow Y'_T$. Since this map is surjective and fiberwisely homeomorphic,

it is a homeomorphism.

$$\begin{array}{ccccc}
F^*Y_T = X' \times_X Y_T & & & & \\
\swarrow p_1 & \searrow p_2 & & & \\
& Y'_T & \xrightarrow{\tilde{F}} & Y_T & \\
& \downarrow p' & & \downarrow p & \\
& X' & \xrightarrow{F} & X &
\end{array}$$

□

Let us consider the following diagram

$$\begin{array}{ccccccc}
K^{\tau'+k}(Y_T) & \longrightarrow & K_T^{\tau'+k}(Y_T) & \xrightarrow{\cong} & K_T^{p^*\tau - \mathbb{P}R_T+k}(Y_T) & \xrightarrow{\otimes[R_T]} & \\
\tilde{F}^* \downarrow & & \tilde{F}^* \downarrow & & \tilde{F}^* \downarrow & & \\
K^{\tilde{F}^*\tau'+k}(Y'_T) & \longrightarrow & K_T^{\tilde{F}^*\tau'+k}(Y'_T) & \xrightarrow{\cong} & K_T^{\tilde{F}^*p^*\tau - \mathbb{P}\tilde{F}^*R_T+k}(Y'_T) & \xrightarrow{\otimes[R'_T]} & \\
& & \xrightarrow{\otimes[R_T]} & K_T^{p^*\tau+k}(Y_T) & \xrightarrow{p!} & K_T^{\tau+k}(X) & \\
& & \tilde{F}^* \downarrow & & F^* \downarrow & & \\
& & \xrightarrow{\otimes[R'_T]} & K_T^{\tilde{F}^*p^*\tau+k}(Y'_T) & \xrightarrow{p'_!} & K_T^{F^*\tau+k}(X'). &
\end{array}$$

The compositions of horizontal sequences are the inverse of $Mackey_{X,T,\tau}$ and $Mackey_{X',T,F^*\tau}$ respectively. If we verify the commutativity of four squares, we obtain the conclusion.

The first is clear.

The second is verified by pull back of isomorphism between two twistings along \tilde{F} .

The third is clear from the following lemma and that F^* preserves \otimes .

Lemma 4.19. *Let R_T and R'_T be the formal K -theory classes associated with (X, τ) and $(X', F^*\tau)$ respectively. Then R'_T is canonically isomorphic to \tilde{F}^*R_T .*

Proof. Let us define an isomorphism as T -equivariant vector bundles $\phi : R'_T \rightarrow F^*R_T$. Let us recall that

$$R_T = \bigcup_{(x,\lambda) \in Y_T} (\mathbb{C}_\lambda)_x, \quad R'_T = \bigcup_{(x',\lambda) \in Y'_T} (\mathbb{C}_\lambda)_{x'},$$

where T_x^τ acts on \mathbb{C}_λ via character $\lambda \in \Lambda_{T,x}^\tau$.

Let us define ϕ as

$$\phi(x', \lambda, z) := (F(x'), \lambda, z),$$

where $x' \in X'$, $\lambda \in \Lambda_{x'}^{F^*\tau}$ and $z \in \mathbb{C}_\lambda$.

Continuity is verified by taking a local trivialization. The isomorphism $R'_T \cong \widetilde{F}^* R_T$ as vector bundles is verified just like the canonical homeomorphism $F^* Y_T \cong Y'_T$. Equivariance is verified as follows.

$$\begin{aligned} t.\phi(x', \lambda, z) &:= t.(F(x'), \lambda, z) \\ &= (F(x'), \lambda, \lambda(t)z) = \phi(x', \lambda, \lambda(t)z), \end{aligned}$$

where $t \in T_{x'}^{F^* \tau} = T_{F(x')}^\tau$. \square

The fourth follows from Lemma 1.38. Let $[\{\mathcal{F}_{(x,\lambda)}\}_{(x,\lambda) \in Y_T}] \in K_T^{p^* \tau + k}(Y_T)$. We can compute $F^* \circ p_!([\{\mathcal{F}_{(x,\lambda)}\}_{(x,\lambda) \in Y_T}])$ and $p'_! \circ \widetilde{F}^*([\{\mathcal{F}_{(x,\lambda)}\}_{(x,\lambda) \in Y_T}])$ as follows.

$$\begin{aligned} F^* \circ p_!([\{\mathcal{F}_{(x,\lambda)}\}_{(x,\lambda) \in Y_T}]) &= F^*([\bigoplus_{\lambda \in \Lambda_{T,x}^\tau} \mathcal{F}_{(x,\lambda)}]_{x \in X}) \\ &= [\bigoplus_{\lambda \in \Lambda_{T,F(x')}^\tau} \mathcal{F}_{(F(x'), \lambda)}]_{x' \in X'} \\ p'_! \circ \widetilde{F}^*([\{\mathcal{F}_{(x,\lambda)}\}_{(x,\lambda) \in Y_T}]) &= p'_!([\{\mathcal{F}_{(F(x'), \lambda)}\}_{(x', \lambda) \in Y'_T}]) \\ &= [\bigoplus_{\lambda \in \Lambda_{T,F(x')}^\tau} \mathcal{F}_{(F(x'), \lambda)}]_{x' \in X'}. \end{aligned}$$

Therefore, the both sides coincide. \square

Let us verify naturality with respect to pull back of group action. However, the sense of “naturality with respect to f^\natural ” is not clear, because $K^{\tau' + k}(Y_T)$ is a non-equivariant twisted K -group. In this paper, we regard the following theorem as naturality with respect to f^\natural .

Theorem 4.20. *Let $p : Y_T \rightarrow X$ and $p' : Y_{T'} \rightarrow X$ be the covering spaces associated with (T, τ) and $(T', f^\natural \tau)$ respectively. The following commutative diagram holds.*

$$\begin{array}{ccc} K^{\tau' + k}(Y_T) & \xrightarrow{\cong} & K_T^{\tau + k}(X) \\ \downarrow \widetilde{t}df_! & & \downarrow f^\natural \\ K^{\tau'' + k}(Y_{T'}) & \xrightarrow{\cong} & K_{T'}^{f^\natural \tau + k}(X) \end{array}$$

where τ' , τ'' and $\widetilde{t}df$ are defined in the proof.

Remark 4.21. *The above theorem is an analogue of the following commutative diagram mentioned in Introduction.*

$$\begin{array}{ccc} K(\Lambda_T) & \xrightarrow{\cong} & R(T) = K_T(\{pt\}) \\ \downarrow ({}^t df)_! & & \downarrow f^\natural \\ K(\Lambda_{T'}) & \xrightarrow{\cong} & R(T') = K_{T'}(\{pt\}) \end{array}$$

Usually, f^\natural is written as f^* which is the pull back of representations along f .

Proof. Let us start from naturality of the covering spaces.

Lemma 4.22. *We can define a continuous fiber map $\widetilde{^t df} : Y_T \rightarrow Y_{T'}$ such that the following diagram holds.*

$$\begin{array}{ccc} Y_T & \xrightarrow{\widetilde{^t df}} & Y_{T'} \\ p \downarrow & & p' \downarrow \\ X & \xlongequal{\quad} & X \end{array}$$

Proof. The family $\{T'_x{}^{f^h\tau}\}_{x \in X}$ of central extensions of T' can be obtained by the fiber product $T'_x{}^{f^h\tau} := T' \times_T T_x^\tau$, that is, the following diagram commutes.

$$\begin{array}{ccc} T'_x{}^{f^h\tau} & \xrightarrow{\widetilde{f}_x} & T_x^\tau \\ \downarrow & & \downarrow \\ T' & \xrightarrow{f} & T \end{array}$$

It can be verified with the same way in the proof of Lemma 4.18.

Since we have a homomorphism $\widetilde{f}_x : T'_x{}^{f^h\tau} \rightarrow T_x^\tau$ induced by f , we have

$$^t d\widetilde{f}_x : \Lambda_{T,x}^\tau \rightarrow \Lambda_{T',x}^{f^h\tau},$$

a pull back of representations. We can define $\widetilde{^t df} : Y_T \rightarrow Y_{T'}$ by $\widetilde{^t df}(x, \lambda) := (x, ^t d\widetilde{f}_x(\lambda))$. Continuity is verified by use of the trivialization defined in Theorem 4.17. \square

Formal K -theory classes $[R_T]$ and $[R_{T'}]$ have the following functoriality.

Lemma 4.23. *Let $[R_T]$ and $[R_{T'}]$ be the formal K -theory classes associated with (T, τ) and $(T', f^h\tau)$ respectively.*

We can define a homomorphism of vector bundles $\widehat{^t df} : R_{T'} \rightarrow R_T$, which makes the following diagram commute.

$$\begin{array}{ccc} R_T & \xrightarrow{\widehat{^t df}} & R_{T'} \\ \downarrow & & \downarrow \\ Y_T & \xrightarrow{\widetilde{^t df}} & Y_{T'} \end{array}$$

Through $f : T' \rightarrow T$, T' acts on R_T . When we regard it as T' -equivariant bundle, $\widetilde{^t df}$ is a homomorphism as T' -equivariant bundles.

Proof. $\widehat{^t df}(x, \lambda', z) := (x, ^t df(\lambda'), z)$ is a homomorphism between two vector bundles. Equivariance is verified by the same method in Lemma 4.19. \square

Let us verify the theorem. Let us consider the following diagram.

$$\begin{array}{ccccccccc}
K^{\tau'+k}(Y_T) & \longrightarrow & K_T^{\tau'+k}(Y_T) & \xrightarrow{\cong} & K_T^{p^*\tau-\mathbb{P}R_T+k}(Y_T) & \xrightarrow{\otimes[R_T]} & K_T^{p^*\tau+k}(Y_T) & \xrightarrow{p!} & K_T^{\tau+k}(X) \\
\parallel & & \downarrow f^\natural & & \downarrow f^\natural & & \downarrow f^\natural & & \downarrow f^\natural \\
(1) & & & (2) & & (3) & & & \\
K^{\tau'+k}(Y_T) & \longrightarrow & K_{T'}^{f^\natural\tau'+k}(Y_T) & \xrightarrow{\cong} & K_{T'}^{f^\natural(p^*\tau-\mathbb{P}R_T)+k}(Y_T) & \xrightarrow{\otimes(\widetilde{t}df)^*[R_{T'}]} & K_{T'}^{f^\natural p^*\tau+k}(Y_T) & \xrightarrow{p'_!} & K_{T'}^{f^\natural\tau+k}(X) \\
\downarrow (\widetilde{t}df)_! & & \downarrow (\widetilde{t}df)_! & & \downarrow (\widetilde{t}df)_! & & \downarrow (\widetilde{t}df)_! & & \downarrow f^\natural \\
(5) & & (6) & & (7) & & & & \\
K^{\tau''+k}(Y_{T'}) & \longrightarrow & K_{T'}^{\tau''+k}(Y_{T'}) & \xrightarrow{\cong} & K_{T'}^{p'^*\tau-\mathbb{P}R_{T'}+k}(Y_{T'}) & \xrightarrow{\otimes[R_{T'}]} & K_{T'}^{p'^*\tau+k}(Y_{T'}) & \xrightarrow{p'_!} & K_{T'}^{f^\natural\tau+k}(X)
\end{array}$$

If we can prove all of the commutativities from (1) to (7), we obtain the required commutativity. Let us notice that $f^\natural p^*\tau = \widetilde{t}df^* p'^* f^\natural \tau$ and $f^\natural(p^*\tau - \mathbb{P}R_T) = \widetilde{t}df^*(p'^* f^\natural \tau - \mathbb{P}R_{T'})$. Therefore, the above push-forward maps are well-defined.

(1), (5) are clear from the definitions of each maps.

(2), (6) follow from that we can pull back the isomorphism between two twistings. τ' and τ'' are defined so that $\widetilde{t}df^* \tau' = f^\natural \tau''$.

(3), (4), (7) follow from the explicit description of these maps. Let us verify (3). Let us notice that $(\widetilde{t}df)^*[R_{T'}] \cong f^\natural[R_T]$.

Let $[\{\mathcal{F}_{(x,\lambda)}\}_{(x,\lambda) \in Y_Y}] \in K_T^{p^*\tau-\mathbb{P}R_T+k}(Y_T)$. We can compute each compositions as follows.

$$\begin{aligned}
f^\natural([\{\mathcal{F}_{(x,\lambda)}\}_{(x,\lambda) \in Y_Y}] \otimes [R_T]) &= f^\natural([\{\mathcal{F}_{(x,\lambda)} \otimes id + \epsilon \otimes \mathbb{C}_\lambda\}_{(x,\lambda) \in Y_Y}]) \\
&= [\{f^\natural(\mathcal{F}_{(x,\lambda)}) \otimes id + \epsilon \otimes \mathbb{C}_{t df(\lambda)}\}_{(x,\lambda) \in Y_Y}]. \\
f^\natural([\{\mathcal{F}_{(x,\lambda)}\}_{(x,\lambda) \in Y_T}] \otimes (\widetilde{t}df)^*[R_{T'}]) &= ([\{f^\natural(\mathcal{F}_{(x,\lambda)})\}_{(x,\lambda) \in Y_T}] \otimes (\widetilde{t}df)^*[R_{T'}]) \\
&= [\{f^\natural(\mathcal{F}_{(x,\lambda)}) \otimes id + \epsilon \otimes \mathbb{C}_{t df(\lambda)}\}_{(x,\lambda) \in Y_T}].
\end{aligned}$$

Therefore, the both sides coincide.

If we notice that

$$(\widetilde{t}df)_!([\{\mathcal{F}_{(x,\lambda)}\}_{(x,\lambda) \in Y_T}]) = [\bigoplus_{t df(\lambda)=\mu} \mathcal{F}_{(x,\lambda)}]_{(x,\mu) \in Y_{T'}},$$

(4) and (7) follow from the same argument. \square

4.1.3 A relationship between index maps and Mackey decomposition

Lemma 4.24. *Let M, Z be manifolds which T acts on trivially and τ be a T -equivariant twisting over M . We suppose that $K^m(Z)$ is free for any $m \in \mathbb{Z}$.*

Let $p : Y_T \rightarrow X$ be the covering space of X associated with (T, τ) . The following commutative diagram holds.

$$\begin{array}{ccc} K_T^{p_1^* \tau + *} (M \times Z) & \xrightarrow{\Phi_{M \times Z}} & K_T^{\tau + *} (M) \otimes K^*(Z) \\ \text{Mackey}_{M \times Z, T, p^* \tau} \downarrow & & \text{Mackey}_{M, T, \tau} \otimes id \downarrow \\ K^{p_1^* \tau' + *} (Y_T \times Z) & \xrightarrow{\Phi_{Y_T \times Z}} & K^{\tau' + *} (Y_T) \otimes K^*(Z) \end{array}$$

Proof. We verify the following commutative diagram.

$$\begin{array}{ccccccc} K^{\tau' + *} (Y_T) \otimes K^*(Z) & \longrightarrow & K_T^{\tau' + *} (Y_T) \otimes K^*(Z) & \xrightarrow{\cong} & K_T^{p^* \tau - \mathbb{P}R + *} (Y_T) \otimes K^*(Z) & \xrightarrow{(\otimes [R]) \otimes id} \\ p_1^* \otimes p_2^* \downarrow & & p_1^* \otimes p_2^* \downarrow & & p_1^* \otimes p_2^* \downarrow & \\ K^{p_1^* \tau' + *} (Y_T \times Z) & \longrightarrow & K_T^{p_1^* \tau' + *} (Y_T \times Z) & \xrightarrow{\cong} & K_T^{p_1^* (p^* \tau - \mathbb{P}R) + *} (Y_T \times Z) & \xrightarrow{\otimes p_1^* [R]} \\ & & \xrightarrow{(\otimes [R]) \otimes id} & & K_T^{p^* \tau + *} (Y_T) \otimes K^*(Z) & \xrightarrow{p_1 \otimes id} K_T^{\tau + *} (M) \otimes K(Z) \\ & & p_1^* \otimes p_2^* \downarrow & & p_1^* \otimes p_2^* \downarrow & \\ & & \xrightarrow{\otimes p_1^* [R]} & & K_T^{p^* \tau + *} (Y_T \times Z) & \xrightarrow{(p \times id)_!} K_T^{p^* \tau + *} (M \times Z) \end{array}$$

where $(\otimes [R]) \otimes id(x \otimes y) = (x \otimes [R]) \otimes y$.

Commutativity of the first square and the fourth one are trivial.

The second one follows by taking the isomorphism between $p_1^* \tau'$ and $p_1^* (p^* \tau - \mathbb{P}R)$ as pull back of the one between τ' and $p^* \tau - \mathbb{P}R$ along p_1 .

The third one is clear from that pull backs preserves \otimes . \square

Let us verify the following theorem. Let us recall that we assume that $K_G^{\tau + n_0 + 1}(M) = 0$, $\dim(Z) = n$, Z has the trivial tangent bundle and $K^m(Z)$ is free for any m in order to define a family index map.

Theorem 4.25. *The following commutative diagram holds.*

$$\begin{array}{ccc} K_T^{p_1^* \tau + n + n_0} (M \times Z) & \xrightarrow{ind_{M \times Z \rightarrow M}} & K_T^{\tau + n_0} (M) \\ \text{Mackey}_{M \times Z, T, p_1^* \tau} \downarrow & & \text{Mackey}_{M, T, \tau} \downarrow \\ K^{p_1^* \tau' + n + n_0} (Y_T \times Z) & \xrightarrow{ind_{Y_T \times Z \rightarrow Y_T}} & K^{\tau' + n_0} (Y_T) \end{array}$$

Proof. The following commutative diagram holds.

$$\begin{array}{ccccc} K_T^{p_1^* \tau + n + n_0} (M \times Z) & \xrightarrow{\Phi_{M \times Z}} & K_T^{\tau + n_0} (M) \otimes K^n(Z) & \xrightarrow{id \otimes ind_Z} & K_T^{\tau + n_0} (M) \\ \text{Mackey}_{M \times Z, T, p_1^* \tau} \downarrow & & \text{Mackey}_{M, T, \tau} \otimes id \downarrow & & \text{Mackey}_{M, T, \tau} \downarrow \\ K^{p_1^* \tau' + n + n_0} (Y_T \times Z) & \xrightarrow{\Phi_{Y_T \times Z}} & K^{\tau' + n_0} (Y_T) \otimes K^n(Z) & \xrightarrow{id \otimes ind_Z} & K^{\tau' + n_0} (Y_T) \end{array}$$

The first square commutes from the above lemma.

The second one commutes clearly. \square

4.2 For tori

Let T be a torus, and τ be a positive central extension of LT and the associated T -equivariant twisting over T . In this section, we construct $f^\#$ for a local injection $f : S \rightarrow T$ so that it has a compatibility with $\text{char}(f)$ by use of the following lemma.

Lemma 4.26. *We have a canonical isomorphism $\text{char}(T, \tau) \cong K(\Lambda_T^\tau / \kappa^\tau(\Pi_T))$.*

Under this identification, if $T = S \times S'$, f is the natural inclusion into the first factor, and $\tau = p_1^ \tau_1 + p_2^* \tau_2$, $\text{char}(f)$ corresponds to the push-forward map $(({}^t df) \text{mod} \Pi_T)_!$, where τ_1 and τ_2 are positive central extension of LS and LS'^\perp respectively. $({}^t df) \text{mod} \Pi_T$ is defined in Lemma 4.28.*

Proof. The canonical isomorphism is defined by defining the values of the canonical generators as $\text{char}(T, \tau) \ni [\lambda]_T \mapsto \delta_{[\lambda]_T} \in K(\Lambda_T^\tau / \kappa^\tau(\Pi_T))$, where $\delta_{[\lambda]_T} \in K(\Lambda_T^\tau / \kappa^\tau(\Pi_T))$ is a trivial vector bundle whose rank is 1 supported on the point $[\lambda]_T$.

As we explain in Lemma 4.28, $\Lambda_T^\tau / \kappa^\tau(\Pi_T) \cong \Lambda_S^{\tau_1} / \kappa^{\tau_1}(\Pi_S) \times \Lambda_{S'}^{\tau_2} / \kappa^{\tau_2}(\Pi_{S'})$ and $({}^t df) \text{mod} \Pi_T$ is the natural projection onto the first factor. By this correspondence and the definition of the push-forward map, the statement about induced homomorphism is clear. \square

We use this identification and the functoriality of push-forward maps, pull backs, family index maps and Mackey decompositions in order to verify our theorem.

We will generalize the correspondence between $(({}^t df) \text{mod} \Pi_T)_!$ and $\text{char}(f)$ in the following.

4.2.1 Direct product

Let T_1, T_2 be tori, $T = T_1 \times T_2$, n_j be the dimension of T_j , τ_j be a positive central extension of LT_j , $i_j : T_j \hookrightarrow T$ be the natural inclusion into the j 'th factor, $p_j : T \rightarrow T_j$ be the natural projection onto the j 'th factor ($j = 1, 2$). We obtain a positive central extension $\tau = p_1^* \tau_1 + p_2^* \tau_2$ of LT from these data. Orientations of T_1 and T_2 naturally define the one of T .

We verify the following theorem.

Theorem 4.27. *If we define as*

$$i_1^\# := \text{ind}_{T_1 \times T_2 \rightarrow T_1} \circ i_1^\natural,$$

the following diagram commutes.

$$\begin{array}{ccc} t.e.K(T, \tau) & \xrightarrow{i_1^\#} & t.e.K(T_1, \tau_1) \\ M.d.T \downarrow & & M.d.T_1 \downarrow \\ \text{char}(T, \tau) & \xrightarrow{\text{char}(i_1)} & \text{char}(T_1, \tau_1) \end{array}$$

Let us start from the lift of $({}^t di_1) \bmod \Pi_T$ to the covering space.

Lemma 4.28. *The following commutative diagram holds.*

$$\begin{array}{ccc} \Lambda_T^\tau \times_{\Pi_T} \mathfrak{t} & \xrightarrow{({}^t di_1 \times dp_1) \bmod \Pi_T} & \Lambda_{T_1}^{\tau_1} \times_{\Pi_{T_1}} \mathfrak{t}_1 \\ \pi \downarrow & & \pi_1 \downarrow \\ \Lambda_T^\tau / \kappa^\tau(\Pi_T) & \xrightarrow{({}^t di_1) \bmod \Pi_T} & \Lambda_{T_1}^{\tau_1} / \kappa^{\tau_1}(\Pi_{T_1}) \end{array}$$

where the horizontal arrows are defined by

$$({}^t di_1) \bmod \Pi_T([\lambda]) := [{}^t di_1(\lambda)]$$

$$({}^t di_1 \times dp_1) \bmod \Pi_T([\lambda, v]) := [({}^t di_1(\lambda), dp_1(v))]$$

and the vertical arrows

$$\pi : \Lambda_T^\tau \times_{\Pi_T} \mathfrak{t} \rightarrow \Lambda_T^\tau / \kappa^\tau(\Pi_T)$$

$$\pi_1 : \Lambda_{T_1}^{\tau_1} \times_{\Pi_{T_1}} \mathfrak{t}_1 \rightarrow \Lambda_{T_1}^{\tau_1} / \kappa^{\tau_1}(\Pi_{T_1})$$

are the trivial vector bundles mentioned in the proof of Corollary 4.14.

Proof. If we verify the well-definedness of $({}^t di_1 \times dp_1) \bmod \Pi_T$, commutativity is clear from the definition. Let $n \in \Pi_T$, $\lambda \in \Lambda_T^\tau$ and $v \in \mathfrak{t}$. We have to verify that ${}^t di_1 \times dp_1(\lambda + \kappa^\tau(n), v + n)$ is equivalent to ${}^t di_1 \times dp_1(\lambda, v)$ under the action of Π_{T_1} .

$$\begin{aligned} & {}^t di_1 \times dp_1(\lambda + \kappa^\tau(n), v + n) = ({}^t di_1(\lambda + \kappa^\tau(n)), dp_1(v + n)) \\ & = ({}^t di_1(\lambda + \kappa^\tau(di_1 \circ dp_1(n) + di_2 \circ dp_2(n))), dp_1(v) + dp_1(n)) \\ & = ({}^t di_1(\lambda) + {}^t di_1 \circ \kappa^\tau \circ di_1(dp_1(n)) + {}^t di_1 \circ \kappa^\tau \circ di_2(dp_2(n)), dp_1(v) + dp_1(n)) \\ & = ({}^t di_1(\lambda) + \kappa^{\tau_1}(dp_1(n)), dp_1(v) + dp_1(n)), \end{aligned}$$

where ${}^t dp_1 \circ \kappa^\tau \circ di_2 = 0$ from Lemma 1.13. \square

Lemma 1.37 implies the following.

Proposition 4.29. *The following commutative diagram holds.*

$$\begin{array}{ccc} K^n(\Lambda_T^\tau \times_{\Pi_T} \mathfrak{t}) & \xrightarrow{(({}^t di_1 \times dp_1) \bmod \Pi_T)_!} & K^{n_1}(\Lambda_{T_1}^{\tau_1} \times_{\Pi_{T_1}} \mathfrak{t}_1) \\ \pi_! \downarrow & & \pi_{1!} \downarrow \\ K(\Lambda_T^\tau / \kappa^\tau(\Pi_T)) & \xrightarrow{({}^t di_1) \bmod \Pi_T)_!} & K(\Lambda_{T_1}^{\tau_1} / \kappa^{\tau_1}(\Pi_{T_1})) \end{array}$$

The lift $({}^t di_1 \times dp_1) \bmod \Pi_T$ of $({}^t di_1) \bmod \Pi_T$ can be written as the composition of the following sequence.

Lemma 4.30. $({}^t di_1 \times dp_1) \text{mod} \Pi_T$ can be written as the composition of the following sequence

$$\Lambda_T^\tau \times_{\Pi_T} \mathfrak{t} \xrightarrow{({}^t di_1 \times id) \text{mod} \Pi_T} \Lambda_{T_1}^{\tau_1} \times_{\Pi_T} \mathfrak{t} \xrightarrow{(id \times dp_1) \text{mod} \Pi_T} \Lambda_{T_1}^{\tau_1} \times_{\Pi_{T_1}} \mathfrak{t}_1,$$

where Π_T acts on $\Lambda_{T_1}^{\tau_1}$ through $dp_1 : \Pi_T \rightarrow \Pi_{T_1}$ and κ^{τ_1} , and

$$({}^t di_1 \times id) \text{mod} \Pi_T([\lambda, v]) := [({}^t di_1(\lambda), v)]$$

$$(id \times dp_1) \text{mod} \Pi_T([\mu, v]) := [(\mu, dp_1(v))].$$

Proof. Well-definedness of these maps can be verified with the same way in the proof of Lemma 4.28. The equality

$$({}^t di_1 \times dp_1) \text{mod} \Pi_T = (id \times dp_1) \text{mod} \Pi_T \circ ({}^t di_1 \times id) \text{mod} \Pi_T$$

follows from that ${}^t di_1 \times dp_1 = (id \times dp_1) \circ ({}^t di_1 \times id)$ □

Through the diffeomorphism $\Lambda_{T_1}^{\tau_1} \times_{\Pi_T} \mathfrak{t} \cong (\Lambda_{T_1}^{\tau_1} \times_{\Pi_{T_1}} \mathfrak{t}_1) \times T_2$, $(id \times dp_1) \text{mod} \Pi_T$ can be thought as the projection onto the first factor.

The following lemma is clear from Theorem 4.20, 4.25. Let us recall that $K_{T_1}^{\tau_1 + \dim(T_1) + 1}(T_1) = 0$ and the tangent bundle of T_2 is trivial. We can trivialize TT_2 via the left translation.

Lemma 4.31. *The following commutative diagram holds.*

$$\begin{array}{ccccc} K_T^{\tau+n}(T) & \xrightarrow{i_1^\sharp} & K_{T_1}^{p_1^* \tau_1 + n}(T_1 \times T_2) & \xrightarrow{ind_{T_1 \times T_2 \rightarrow T_1}} & K_{T_1}^{\tau_1 + n_1}(T_1) \\ \text{Mackey}_{T, T, \tau} \downarrow & & \text{Mackey}_{T_1 \times T_2, T_1, p_1^* \tau} \downarrow & & \text{Mackey}_{T_1, T_1, \tau_1} \downarrow \\ K^n(Y_T) & \xrightarrow{({}^t \widetilde{di_1})_!} & K^n(Y_{T_1} \times T_2) & \xrightarrow{ind_{Y_{T_1} \times T_2 \rightarrow Y_{T_1}}} & K^{n_1}(Y_{T_1}) \end{array}$$

where $n := n_1 + n_2$, $Y_T := \Lambda_T^\tau \times_{\Pi_T} \mathfrak{t}$ and $Y_{T_1} := \Lambda_{T_1}^{\tau_1} \times_{\Pi_{T_1}} \mathfrak{t}_1$.

The decomposition of $({}^t di_1 \times dp_1) \text{mod} \Pi_T$ and Theorem 1.38 imply the following lemma.

Lemma 4.32. *The following commutative diagram holds.*

$$\begin{array}{ccccc} K^n(Y_T) & \xrightarrow{({}^t \widetilde{di_1})_!} & K^n(Y_{T_1} \times T_2) & \xrightarrow{ind_{Y_{T_1} \times T_2 \rightarrow Y_{T_1}}} & K^{n_1}(Y_{T_1}) \\ \downarrow \pi_! & & & & \downarrow \pi_{1!} \\ K(\Lambda_T^\tau / \kappa^\tau(\Pi_T)) & \xrightarrow{({}^t di_1) \text{mod} \Pi_T)_!} & & & K(\Lambda_{T_1}^{\tau_1} / \kappa^{\tau_1}(\Pi_{T_1})) \end{array}$$

Let us notice that the upper sequence in the above lemma is the lower one in Lemma 4.31.

Let us verify the theorem.

Proof of Theorem 4.27. Let us recall Lemma 4.26, that is, the following commutative diagram holds.

$$\begin{array}{ccc}
K(\Lambda_T^\tau / \kappa^\tau(\Pi_T)) & \xrightarrow{(({}^t di_1) \text{mod} \Pi_T)_!} & K(\Lambda_{T_1}^{\tau_1} / \kappa^{\tau_1}(\Pi_{T_1})) \\
\cong \downarrow & & \cong \downarrow \\
\text{char}(T, \tau) & \xrightarrow{\text{char}(i_1)} & \text{char}(T_1, \tau_1)
\end{array}$$

Combining with Lemma 4.32, the following commutative diagram holds.

$$\begin{array}{ccccc}
K^n(Y_T) & \xrightarrow{(\widetilde{{}^t di_1})_!} & K^n(Y_{T_1} \times T_2) & \xrightarrow{\text{ind}_{Y_{T_1} \times T_2 \rightarrow Y_{T_1}}} & K^{n_1}(Y_{T_1}) \\
\downarrow \pi_! & & & & \downarrow \pi_! \\
K(\Lambda_T^\tau / \kappa^\tau(\Pi_T)) & \xrightarrow{(({}^t di_1) \text{mod} \Pi_T)_!} & & & K(\Lambda_{T_1}^{\tau_1} / \kappa^{\tau_1}(\Pi_{T_1})) \\
\cong \downarrow & & & & \cong \downarrow \\
\text{char}(T, \tau) & \xrightarrow{\text{char}(i_1)} & & & \text{char}(T_1, \tau_1)
\end{array}$$

Let us consider the following diagram.

$$\begin{array}{ccccc}
K_T^{\tau+n}(T) & \xrightarrow{i_1^\natural} & K_{T_1}^{p_1^* \tau_1+n}(T) & \xrightarrow{\text{ind}_{T_1 \times T_2 \rightarrow T_1}} & K_{T_1}^{\tau_1+n_1}(T_1) \\
\downarrow \text{Mackey}_{T,T,\tau} & & \downarrow \text{Mackey}_{T,T_1,p_1^* \tau_1} & & \downarrow \text{Mackey}_{T_1,T_1,\tau_1} \\
K^n(Y_T) & \xrightarrow{(\widetilde{{}^t di_1})_!} & K^n(Y_{T_1} \times T_2) & \xrightarrow{\text{ind}_{Y_{T_1} \times T_2 \rightarrow Y_{T_1}}} & K^{n_1}(Y_{T_1}) \\
\downarrow \pi_! & & & & \downarrow \pi_! \\
\text{char}(T, \tau) & \xrightarrow{\text{char}(i_1)} & & & \text{char}(T_1, \tau_1)
\end{array}$$

Let us recall that $M.d.T = \pi_! \circ \text{Mackey}_{T,T,\tau}$ and $M.d.T_1 = \pi_! \circ \text{Mackey}_{T_1,T_1,\tau_1}$.

(1) commutes from Theorem 4.20. (2) commutes from Theorem 4.25. (3) commutes from the above commutative diagram.

□

4.2.2 Finite covering

Let $q : T' \rightarrow T$ be a finite covering of an n dimensional torus and τ be a positive central extension of LT and the associated T -equivariant twisting over T . In this section, we verify the following theorem.

Theorem 4.33. *If we define*

$$q^\# := q^* \circ q^\natural$$

the following commutative diagram holds.

$$\begin{array}{ccc} t.e.K(T, \tau) & \xrightarrow{q^\#} & t.e.K(T', q^*\tau) \\ M.d.T \downarrow & & M.d.T' \downarrow \\ char(T, \tau) & \xrightarrow{char(q)} & char(T', q^*\tau) \end{array}$$

Remark 4.34. $q^\#$ is the composition of

$$K_T^{\tau+\dim(T)}(T) \xrightarrow{q^\natural} K_{T'}^{q^\natural\tau+\dim(T)}(T) \xrightarrow{q^*} K_{T'}^{q^*\tau+\dim(T')}(T').$$

It is written as simply q^* in [FHT1]. In the terminology of groupoids, q determines a functor $(q, q) : T'/T' \rightarrow T/T$ and q^* is the pull back along the functor (q, q) .

Let us compute $K_{T'}^{q^\natural\tau+\dim(T)}(T)$ by use of a Mackey decomposition.

Theorem 4.35.

$$K_{T'}^{q^\natural\tau+k}(T) \cong \begin{cases} \mathbb{Z}[\Lambda_{T'}^{q^*\tau}/{}^t dq(\kappa^\tau(\Pi_T))] & k = \dim(T) \pmod{2} \\ 0 & k = \dim(T) + 1 \pmod{2} \end{cases}$$

Proof. From the definition of $q^\natural\tau$, if one travels on T along $n \in \Pi_T$, the character $\lambda \in \Lambda_{T'}^{q^\natural\tau}$ changes to $\lambda + {}^t dq(\kappa^\tau(n))$. Therefore, the covering space constructed in Section 4.1.2 is

$$p^{(\cdot)} : \Lambda_{T'}^{q^*\tau} \times_{\Pi_T} \mathfrak{t} \rightarrow T,$$

where Π_T acts on $\Lambda_{T'}^{q^*\tau}$ via ${}^t dq \circ \kappa^\tau$ and $\Lambda_{T'} \curvearrowright \Lambda_{T'}^{q^*\tau}$. It has a structure of a trivial vector bundle

$$\pi^{(\cdot)} : \Lambda_{T'}^{q^*\tau} \times_{\Pi_T} \mathfrak{t} \rightarrow \Lambda_{T'}^{q^*\tau}/{}^t dq(\kappa^\tau(\Pi_T)).$$

Through Thom isomorphism $\pi_!^{(\cdot)}$, we obtain the conclusion. \square

Lemma 4.36. Let us consider the maps

$$\Lambda_T^\tau/\kappa^\tau(\Pi_T) \xrightarrow{({}^t dq) \bmod \Pi_T} \Lambda_{T'}^{q^*\tau}/{}^t dq(\kappa^\tau(\Pi_T)) \xleftarrow{r} \Lambda_{T'}^{q^*\tau}/\kappa^{q^*\tau}(\Pi_{T'}),$$

where

$$\begin{aligned} ({}^t dq) \bmod \Pi_T([\lambda]_T) &:= [{}^t dq(\lambda)] \\ r([\mu]_{T'}) &:= [\mu]. \end{aligned}$$

$[\mu]$ is the Π_T -orbit of μ in $\Lambda_{T'}^{q^*\tau}$.

Then $char(q)$ corresponds to $r^* \circ (({}^t dq) \bmod \Pi_T)_!$. That is, the following commutative diagram holds.

$$\begin{array}{ccc} K(\Lambda_T^\tau/\kappa^\tau(\Pi_T)) & \xrightarrow{(({}^t dq) \bmod \Pi_T)_!} & K(\Lambda_{T'}^{q^*\tau}/{}^t dq(\kappa^\tau(\Pi_T))) \xrightarrow{r^*} K(\Lambda_{T'}^{q^*\tau}/\kappa^{q^*\tau}(\Pi_{T'})) \\ \downarrow \cong & & \downarrow \cong \\ char(T, \tau) & \xrightarrow{char(q)} & char(T', q^*\tau) \end{array}$$

Proof. Since $({}^t dq) \bmod \Pi_T$ is an injection, for $\lambda \in \Lambda_T^\tau$,

$$(({}^t dq) \bmod \Pi_T)_!(\delta_{[\lambda]_T}) = \delta_{[{}^t dq(\lambda)]}.$$

On the other hand, for $\lambda' \in \Lambda_{T'}^{q^* \tau}$,

$$r^*(\delta_{[\lambda']}) = \sum_{n \in \Pi_T / dq(\Pi_{T'})} \delta_{[\lambda' + {}^t dq(\kappa^\tau(n))]_{T'}}.$$

If we notice that

$$r^{-1}(\{[\lambda']\}) = \{[\lambda' + {}^t dq(\kappa^\tau(n))]_{T'} \mid n \in \Pi_T / dq(\Pi_{T'})\},$$

the above computation is clear from the definition of pull back of vector bundles.

As we compute at Lemma 3.4, the above commutative diagram holds. \square

Let us lift r to the covering spaces. We can verify the following lemma with the same way in Lemma 4.28.

Lemma 4.37. *The following commutative diagram holds.*

$$\begin{array}{ccc} \Lambda_{T'}^{q^* \tau} \times_{\Pi_T} \mathfrak{t} & \xleftarrow{(id \times dq) \bmod \Pi_{T'}} & \Lambda_{T'}^{q^* \tau} \times_{\Pi_{T'}} \mathfrak{t}' \\ \pi^{(\cdot)} \downarrow & & \pi' \downarrow \\ \Lambda_{T'}^{q^* \tau} / {}^t dq(\kappa^\tau(\Pi_T)) & \xleftarrow{r} & \Lambda_{T'}^{q^* \tau} / \kappa^{q^* \tau}(\Pi_{T'}) \end{array}$$

where $\pi' : \Lambda_{T'}^{q^* \tau} \times_{\Pi_{T'}} \mathfrak{t}' \rightarrow \Lambda_{T'}^{q^* \tau} / \kappa^{q^* \tau}(\Pi_{T'})$ is the trivial vector bundle defined in the proof of Theorem 4.35, and $(id \times dq) \bmod \Pi_{T'}([\lambda, v]) := [(\lambda, dq(v))]$.

From the above diagram, the vector bundle $\Lambda_{T'}^{q^* \tau} \times_{\Pi_{T'}} \mathfrak{t}'$ is the pull back of $\Lambda_{T'}^{q^* \tau} \times_{\Pi_T} \mathfrak{t}$ along r .

Let us recall that push-forward is given by the inverse of the tensor product with Thom class. Combining the above lemma with naturality of Thom class, we obtain the following.

Proposition 4.38. *The following commutative diagram holds.*

$$\begin{array}{ccc} K^n(\Lambda_{T'}^{q^* \tau} \times_{\Pi_T} \mathfrak{t}) & \xrightarrow{((id \times dq) \bmod \Pi_{T'})^*} & K^n(\Lambda_{T'}^{q^* \tau} \times_{\Pi_{T'}} \mathfrak{t}') \\ \pi_!^{(\cdot)} \downarrow & & \pi'_! \downarrow \\ K(\Lambda_{T'}^{q^* \tau} / {}^t dq(\kappa^\tau(\Pi_T))) & \xrightarrow{r^*} & K(\Lambda_{T'}^{q^* \tau} / \kappa^{q^* \tau}(\Pi_{T'})) \end{array}$$

where $(id \times dq) \bmod \Pi_{T'}([\mu, w]) := [(\mu, dq(w))]$.

Let us lift $({}^t dq) \bmod \Pi_T$ to the covering spaces.

The following lemma can be verified with the same way in Lemma 4.28.

Lemma 4.39. *The following diagram commutes.*

$$\begin{array}{ccc}
\Lambda_T^\tau \times_{\Pi_T} \mathfrak{t} & \xrightarrow{({}^t dq \times id) \bmod \Pi_T} & \Lambda_{T'}^{q^* \tau} \times_{\Pi_T} \mathfrak{t} \\
\pi \downarrow & & \pi^{(\prime)} \downarrow \\
\Lambda_T^\tau / \kappa^\tau(\Pi_T) & \xrightarrow{({}^t dq) \bmod \Pi_T} & \Lambda_{T'}^{q^* \tau} / {}^t dq(\kappa^\tau(\Pi_T))
\end{array}$$

From Lemma 1.37, we can verify the following.

Proposition 4.40. *The following commutative diagram holds.*

$$\begin{array}{ccc}
K^n(\Lambda_T^\tau \times_{\Pi_T} \mathfrak{t}) & \xrightarrow{(({}^t dq \times id) \bmod \Pi_T)_!} & K^n(\Lambda_{T'}^{q^* \tau} \times_{\Pi_T} \mathfrak{t}) \\
\pi_! \downarrow & & \pi_!^{(\prime)} \downarrow \\
K(\Lambda_T^\tau / \kappa^\tau(\Pi_T)) & \xrightarrow{({}^t dq) \bmod \Pi_T)_!} & K(\Lambda_{T'}^{q^* \tau} / {}^t dq(\kappa^\tau(\Pi_T)))
\end{array}$$

Let us verify that $(id \times dq) \bmod \Pi_{T'}$ is the lift of q , that is, if we use the notation in Theorem 4.17, $(id \times dq) \bmod \Pi_{T'}$ can be written as \tilde{q} .

Lemma 4.41. *The following commutative diagram holds.*

$$\begin{array}{ccc}
\Lambda_{T'}^{q^* \tau} \times \mathfrak{t}' & \xrightarrow{id \times dq} & \Lambda_{T'}^{q^* \tau} \times \mathfrak{t} \\
\overline{p'} \downarrow & & \overline{p^{(\prime)}} \downarrow \\
\Lambda_{T'}^{q^* \tau} \times_{\Pi_{T'}} \mathfrak{t}' & \xrightarrow{(id \times dq) \bmod \Pi_{T'}} & \Lambda_{T'}^{q^* \tau} \times_{\Pi_T} \mathfrak{t} \\
p' \downarrow & & p^{(\prime)} \downarrow \\
T' & \xrightarrow{q} & T
\end{array}$$

where $\overline{p'}$ and $\overline{p^{(\prime)}}$ are the natural projections.

Proof. That $p^{(\prime)} \circ \overline{p^{(\prime)}} \circ (id \times dq) = q \circ p' \circ \overline{p'}$ follows from the property of the tangent map dq , and that $\overline{p^{(\prime)}} \circ (id \times dq) = (id \times dq) \bmod \Pi_T \circ \overline{p'}$ follows from the definition of $(id \times dq) \bmod \Pi_{T'}$. Therefore, that $p^{(\prime)} \circ (id \times dq) \bmod \Pi_{T'} = q \circ p'$ holds. \square

This lemma and Theorem 4.17 imply the following. Let n be the dimension of T .

Proposition 4.42. *$((id \times dq) \bmod \Pi_{T'})^*$ corresponds to q^* under the Mackey decomposition. That is, the following commutative diagram holds.*

$$\begin{array}{ccc}
K_{T'}^{q^* \tau + n}(T) & \xrightarrow{q^*} & K_{T'}^{q^* \tau + n}(T') \\
\text{Mackey}_{T, T', q^* \tau} \downarrow & & \downarrow \text{Mackey}_{T', T', q^* \tau} \\
K^n(\Lambda_{T'}^{q^* \tau} \times_{\Pi_T} \mathfrak{t}) & \xrightarrow{((id \times dq) \bmod \Pi_{T'})^*} & K^n(\Lambda_{T'}^{q^* \tau} \times_{\Pi_{T'}} \mathfrak{t}')
\end{array}$$

Let us verify the theorem.

Proof of Theorem 4.33. Let us consider the following diagram.

$$\begin{array}{ccccc}
K_T^{\tau+n}(T) & \xrightarrow{q^\natural} & K_{T'}^{q^\natural\tau+n}(T) & \xrightarrow{q^*} & K_{T'}^{q^*\tau+n}(T') \\
\downarrow \text{Mackey}_{T,T,\tau} \quad (1) & & \downarrow \text{Mackey}_{T,T',q^\natural\tau} \quad (2) & & \downarrow \text{Mackey}_{T',T',q^*\tau} \\
K^n(\Lambda_T^\tau \times_{\Pi_T} \mathfrak{t}) & \xrightarrow{((^t dq \times id) \text{mod } \Pi_T)_!} & K^n(\Lambda_{T'}^{q^*\tau} \times_{\Pi_T} \mathfrak{t}) & \xrightarrow{((id \times dq) \text{mod } \Pi_{T'})^*} & K^n(\Lambda_{T'}^{q^*\tau} \times_{\Pi_{T'}} \mathfrak{t}') \\
\downarrow \pi_! \quad (3) & & \downarrow \pi_!^{(')} \quad (4) & & \downarrow \pi'_! \\
K(\Lambda_T^\tau / \kappa^\tau(\Pi_T)) & \xrightarrow{((^t dq) \text{mod } \Pi_T)_!} & K(\Lambda_{T'}^{q^*\tau} / {}^t dq(\kappa^\tau(\Pi_T))) & \xrightarrow{r^*} & K(\Lambda_{T'}^{q^*\tau} / \kappa^{q^*\tau}(\Pi_{T'}))
\end{array}$$

- (1) commutes from Theorem 4.20. (2) commutes from Lemma 4.41 and 4.17.
(3) commutes from Proposition 4.40. (4) commutes from Proposition 4.38.
From Lemma 4.36, we obtain the conclusion.

□

4.2.3 Local injection

Let S and T be tori, $f : S \rightarrow T$ be a local injection and τ be a positive central extension. Let us recall that $t.e.K(T, \tau)$ is defined by $K_T^{\tau+dim(T)}(T)$.

From Theorem 3.10, f can be decomposed as follows.

$$S \xrightarrow{q} S / \ker(f) \xrightarrow{i_1} S / \ker(f) \times S^\perp \xrightarrow{f \cdot j} T,$$

where q is the natural finite covering, i_1 is the natural inclusion into the first factor, and $j : S^\perp \rightarrow T$ is the natural inclusion.

Let us verify the following theorem.

Theorem 4.43. *If we define as*

$$f^\# := q^\# \circ i_1^\# \circ (f \cdot j)^\#,$$

the following commutative diagram holds.

$$\begin{array}{ccc}
t.e.K(T, \tau) & \xrightarrow{f^\#} & t.e.K(S, f^*\tau) \\
M.d.T \downarrow & & M.d.S \downarrow \\
char(T, \tau) & \xrightarrow{char(f)} & char(S, f^*\tau)
\end{array}$$

Proof. The following commutative diagram holds from Theorem 4.27 and Theorem 4.33.

$$\begin{array}{ccccc}
t.e.K(T, \tau) & \xrightarrow{(f \cdot j)^\#} & t.e.K(S / \ker(f) \times S^\perp, (f \cdot j)^*\tau) & \xrightarrow{i_1^\#} & \\
M.d.T \downarrow & & M.d.S / \ker(f) \times S^\perp \downarrow & & \\
char(T, \tau) & \xrightarrow{char(f \cdot j)} & char(S / \ker(f) \times S^\perp, (f \cdot j)^*\tau) & \xrightarrow{char(i_1)} &
\end{array}$$

$$\begin{array}{ccccc}
i_1^\# & \longrightarrow & t.e.K(S/\ker(f), i_1^*(f \cdot j)^*\tau) & \xrightarrow{q^\#} & t.e.K(S, f^*\tau) \\
& & \downarrow M.d.S/\ker(f) & & \downarrow M.d.S \\
\text{char}(i_1) & \longrightarrow & \text{char}(S/\ker(f), i_1^*(f \cdot j)^*\tau) & \xrightarrow{\text{char}(q)} & \text{char}(S, f^*\tau)
\end{array}$$

Form Theorem 3.12,

$$\text{char}(f) = \text{char}(q) \circ \text{char}(i_1) \circ \text{char}(f \cdot j),$$

and by the definition,

$$f^\# = q^\# \circ i_1^\# \circ (f \cdot j)^\#.$$

Therefore, we obtain the conclusion. \square

4.3 For compact connected G with torsion-free π_1

Let us extend the above construction to compact connected Lie groups with torsion-free π_1 .

Let G and H be compact connected Lie groups with torsion-free π_1 , τ be a positive central extension of LG and the associated G -equivariant twisting over G , f satisfy the decomposable condition, and S and T be chosen maximal tori of H and G respectively such that $f(S) \subseteq T$. That is, the following commutative diagram holds.

$$\begin{array}{ccc}
H & \xrightarrow{f} & G \\
i \uparrow & & \uparrow k \\
S & \xrightarrow{f} & T
\end{array}$$

In this section, we use the same character for the restriction of homomorphisms to subgroup or induced map to the quotient group. For example, $f : H \rightarrow G$ determines group homomorphisms $f : H/\ker(f) \rightarrow G$ and $f : S \rightarrow T$.

Let us recall the result in [FHT1].

Theorem 4.44 ([FHT1] Theorem 4.27). *If f is an injection and $\text{rank}(G) = \text{rank}(H)$, the following commutative diagram holds.*

$$\begin{array}{ccc}
K_G^{\tau + \text{rank}(G)}(G) & \xrightarrow{f^* \circ f^\natural} & K_H^{f^* \tau + \text{rank}(H)} \\
M.d.G \downarrow & & \downarrow M.d.H \\
\text{char}(G, \tau) & \xrightarrow{\text{char}(f)} & \text{char}(H, f^* \tau)
\end{array}$$

This theorem tells us that $f^\#$ has been defined and holds naturality with $\text{char}(f)$. Let us extend the result for more general cases. That is, we verify the following theorem.

Theorem 4.45. *We can define $f^\#$ for f satisfying the decomposable condition and the following commutative diagram holds.*

$$\begin{array}{ccc} t.e.K(G, \tau) & \xrightarrow{f^\#} & t.e.K(H, f^*\tau) \\ M.d.G \downarrow & & M.d.H \downarrow \\ char(G, \tau) & \xrightarrow{char(f)} & char(H, f^*\tau) \end{array}$$

4.3.1 The decomposition of a homomorphism satisfying the decomposable condition

As we explain in the proof of Theorem 3.20, $f : H \rightarrow G$ can be written as the composition of the following series

$$H \xrightarrow{q} H/\ker(f) \xrightarrow{i_1} H/\ker(f) \times S^\perp \xrightarrow{f \cdot j} G$$

if f satisfies the decomposable condition. By restricting this sequence to maximal tori, we obtain the following commutative diagram

$$\begin{array}{ccccccc} H & \xrightarrow{q} & H/\ker(f) & \xrightarrow{i_1} & H/\ker(f) \times S^\perp & \xrightarrow{f \cdot j} & G \\ i \uparrow & & i \uparrow & & i \times id \uparrow & & k \uparrow \\ S & \xrightarrow{q} & S/\ker(f) & \xrightarrow{i_1} & S/\ker(f) \times S^\perp & \xrightarrow{f \cdot j} & T \end{array} \quad \begin{array}{c} \swarrow j \\ S^\perp \end{array}$$

Moreover, by the definition of S^\perp , $(f \cdot j)^*\tau$ can be written as

$$p_1^*\tau_{H/\ker(f)} + p_2^*\tau_{S^\perp},$$

where $\tau_{H/\ker(f)} = ((f \cdot j) \circ i_1)^*\tau$ and $\tau_{S^\perp} = j^*\tau$.

Let us notice that the restriction of q and $f \cdot j$ to maximal tori are finite coverings, and the one of i_1 is the natural inclusion into the first factor of the direct product. By use of theorems of the previous section, we construct $f^\#$ for f .

4.3.2 “Direct product”

Let H be a compact connected Lie group with torsion-free π_1 , S be a maximal torus of H , S^\perp be a torus, and τ_H and τ_{S^\perp} be positive central extensions of LH and LS^\perp respectively. $i_1 : H \rightarrow H \times S^\perp$, $p_1 : H \times S^\perp \rightarrow H$ and $p_2 : H \times S^\perp \rightarrow S^\perp$ are as usual. Then, $\tau := p_1^*\tau_H + p_2^*\tau_{S^\perp}$ is a positive central extension of $L(H \times S^\perp)$ and the associated $H \times S^\perp$ -equivariant twisting over $H \times S^\perp$.

Let us define

$$i_1^\# : t.e.K(H \times S^\perp, \tau) \rightarrow t.e.K(H, \tau_H)$$

by use of the family index map like the case of tori.

Theorem 4.46. *If we define as*

$$i_1^\# := \text{ind}_{H \times S^\perp \rightarrow H} \circ i_1^\natural,$$

the following commutative diagram holds.

$$\begin{array}{ccc} t.e.K(H \times S^\perp, \tau) & \xrightarrow{i_1^\#} & t.e.K(H, \tau_H) \\ M.d._{H \times S^\perp} \downarrow & & M.d._H \downarrow \\ \text{char}(H \times S^\perp, \tau) & \xrightarrow{\text{char}(i_1)} & \text{char}(H, \tau_H) \end{array}$$

Let us start from the reduction to maximal tori. $n := \text{rank}(H)$ and $m := \dim(S^\perp)$.

Proposition 4.47. *The following commutative diagram holds.*

$$\begin{array}{ccc} t.e.K(H \times S^\perp, \tau) & \xrightarrow{(i \times id)^\#} & t.e.K(S \times S^\perp, (i \times id)^* \tau) \\ i_1^\# \downarrow & & i_1^\# \downarrow \\ t.e.K(H, \tau_H) & \xrightarrow{i^\#} & t.e.K(S, i^* \tau_H) \end{array}$$

where $(i \times id)^\#$ and $i^\#$ have been defined in Theorem 4.44.

Proof.

$$\begin{array}{ccccc} K_{H \times S^\perp}^{\tau+n+m}(H \times S^\perp) & \xrightarrow{(i \times id)^\natural} & K_{S \times S^\perp}^{(i \times id)^\natural \tau+n+m}(H \times S^\perp) & \xrightarrow{(i \times id)^*} & K_{S \times S^\perp}^{(i \times id)^* \tau+n+m}(S \times S^\perp) \\ \downarrow i_1^\natural & (1) & \downarrow i_1^\natural & (2) & \downarrow i_1^\natural \\ K_H^{p_1^* \tau_H+n+m}(H \times S^\perp) & \xrightarrow{i^\natural} & K_S^{p_1^* i^\natural \tau_H+n+m}(H \times S^\perp) & \xrightarrow{(i \times id)^*} & K_S^{i^* \tau_H+n+m}(S \times S^\perp) \\ \downarrow \Phi_{H \times S^\perp, H, \tau_H} & (3) & \downarrow \Phi_{H \times S^\perp, S, i^\natural \tau_H} & (4) & \downarrow \Phi_{S \times S^\perp, S, i^* \tau_H} \\ K_H^{\tau_H+n}(H) \otimes K^m(S^\perp) & \xrightarrow{i^\natural \otimes id} & K_S^{i^\natural \tau_H+n}(H) \otimes K^m(S^\perp) & \xrightarrow{i^* \otimes id} & K_S^{i^* \tau_H+n}(S) \otimes K^m(S^\perp) \\ \downarrow id \otimes ind_{S^\perp} & (5) & \downarrow id \otimes ind_{S^\perp} & (6) & \downarrow id \otimes ind_{S^\perp} \\ K_H^{\tau_H+n}(H) & \xrightarrow{i^\natural} & K_S^{i^\natural \tau_H+n}(H) & \xrightarrow{i^*} & K_S^{i^* \tau_H+n}(S) \end{array}$$

(1), (2), (5) and (6) clearly commute.

(3) and (4) commute from Lemma 4.10 and 4.9 respectively. \square

Proof of Theorem 4.46. Let us consider the following diagram.

Let $G := H \times S^\perp$, $T := S \times S^\perp$ and $k := i \times id : S \times S^\perp = T \hookrightarrow H \times S^\perp = G$.

$$\begin{array}{ccccc}
t.e.K(G, \tau) & \xrightarrow{i_1^\#} & t.e.K(H, \tau_H) & & \\
\downarrow k^\# & \searrow M.d.G & \xrightarrow{(1)} & \swarrow M.d.H & \downarrow i^\# \\
& char(G, \tau) & \xrightarrow{char(i_1)} & char(H, \tau_H) & \\
& \downarrow char(k) & (3) & \downarrow char(i) & (4) \\
& char(T, k^*\tau) & \xrightarrow{char(i_1)} & char(S, i^*\tau_H) & \\
& \swarrow M.d.T & (5) & \swarrow M.d.S & \\
t.e.K(T, k^*\tau) & \xrightarrow{i_1^\#} & t.e.K(S, i^*\tau_H) & &
\end{array}$$

We want to verify the commutativity of (1). Since $char(i)$ is injective, this is equivalent to

$$char(i) \circ M.d.H \circ i_1^\# = char(i) \circ char(i_1) \circ M.d.G.$$

The followings have been verified.

(2) and (4) follow from Theorem 4.44.

(3) follows from the definition of $char(i_1)$.

(5) follows from Theorem 4.27.

That $i^\# \circ i_1^\# = i_1^\# \circ k^\#$ follows from the above lemma (the biggest square).

The following computation implies the theorem.

$$\begin{aligned}
char(i) \circ M.d.H \circ i_1^\# &= M.d.S \circ i^\# \circ i_1^\# \\
&= M.d.S \circ i_1^\# \circ k^\# = char(i_1) \circ M.d.T \circ k^\# \\
&= char(i_1) \circ char(k) \circ M.d.G \\
&= char(i) \circ char(i_1) \circ M.d.G.
\end{aligned}$$

□

4.3.3 “Finite covering”

Let H, G be compact connected Lie group with torsion-free π_1 , τ be a positive central extension of LG and the associated G -equivariant twisting over G , and $f : H \rightarrow G$ be a smooth group homomorphism satisfying the decomposable condition such that the restriction of it to maximal torus is a finite covering. Let S and T be maximal tori of H and G respectively such that $f(S) \subseteq T$. That is, the following commutative diagram holds.

$$\begin{array}{ccc}
H & \xrightarrow{f} & G \\
\uparrow i & & \uparrow k \\
S & \xrightarrow{f} & T
\end{array}$$

We verify the following theorem.

Theorem 4.48. *If we define as*

$$f^\# := f^* \circ f^\natural,$$

the following commutative diagram holds.

$$\begin{array}{ccc} t.e.K(G, \tau) & \xrightarrow{f^\#} & t.e.K(H, f^*\tau) \\ M.d.G \downarrow & & M.d.H \downarrow \\ char(G, \tau) & \xrightarrow{char(f)} & char(H, f^*\tau) \end{array}$$

Remark 4.49. *In Theorem 4.44, the restriction of f to maximal torus is supposed to be injective.*

Proof. Let us start from the reduction to maximal tori. $n := rank(H) = rank(G)$.

Proposition 4.50. *The following commutative diagram holds.*

$$\begin{array}{ccc} t.e.K(G, \tau) & \xrightarrow{f^\#} & t.e.K(H, f^*\tau) \\ k^\# \downarrow & & i^\# \downarrow \\ t.e.K(T, k^*\tau) & \xrightarrow{f^\#} & t.e.K(S, i^*f^*\tau) \end{array}$$

Proof. It follows from the following commutative diagram.

$$\begin{array}{ccccc} K_G^{\tau+n}(G) & \xrightarrow{f^\natural} & K_H^{f^\natural\tau+n}(G) & \xrightarrow{f^*} & K_H^{f^*\tau+n}(H) \\ k^\natural \downarrow & & i^\natural \downarrow & & i^\natural \downarrow \\ K_T^{k^\natural\tau+n}(G) & \xrightarrow{f^\natural} & K_S^{i^\natural f^\natural\tau+n}(G) & \xrightarrow{f^*} & K_S^{i^\natural f^*\tau+n}(H) \\ k^* \downarrow & & k^* \downarrow & & i^* \downarrow \\ K_T^{k^*\tau+n}(T) & \xrightarrow{f^\natural} & K_S^{f^\natural k^*\tau+n}(T) & \xrightarrow{f^*} & K_S^{i^* f^*\tau+n}(S) \end{array}$$

□

Let us consider the following diagram just like the proof of Theorem 4.46.

$$\begin{array}{ccccc}
t.e.K(G, \tau) & \xrightarrow{f^\#} & t.e.K(H, f^*\tau) & & \\
\downarrow k^\# & \nearrow M.d.G & \searrow M.d.H & & \downarrow i^\# \\
& char(G, \tau) & \xrightarrow{char(f)} & char(H, f^*\tau) & \\
& \downarrow char(k) & & \downarrow char(i) & \\
& char(T, k^*\tau) & \xrightarrow{char(f)} & char(S, f^*k^*\tau) & \\
\uparrow M.d.T & & & & \uparrow M.d.S \\
t.e.K(T, k^*\tau) & \xrightarrow{f^\#} & t.e.K(S, f^*k^*\tau) & &
\end{array}$$

Verified commutativity is represented by “ \circlearrowright ”. Moreover, we have known the commutativity of the biggest square already in the above lemma.

We want to verify that $M.d.H \circ f^\# = char(f) \circ M.d.G$. It can be verified with the same way in the case of Theorem 4.46. \square

4.3.4 Proof of Theorem 4.45.

Let $f : H \rightarrow G$ be a smooth group homomorphism satisfying the decomposable condition, τ be a positive central extension of LG . S and T are chosen maximal tori such that $f(S) \subseteq T$.

Let us recall that we can define the orthogonal completion torus $S^\perp \subseteq T$ and decompose f as the following sequence

$$H \xrightarrow{q} H/\ker(f) \xrightarrow{i_1} H/\ker(f) \times S^\perp \xrightarrow{f \cdot j} G$$

By restricting this sequence to maximal tori, we obtain the following commutative diagram

$$\begin{array}{ccccccc}
H & \xrightarrow{q} & H/\ker(f) & \xrightarrow{i_1} & H/\ker(f) \times S^\perp & \xrightarrow{f \cdot j} & G \\
\uparrow i & & \uparrow i & & \uparrow i \times id & & \uparrow k \\
S & \xrightarrow{q} & S/\ker(f) & \xrightarrow{i_1} & S/\ker(f) \times S^\perp & \xrightarrow{f \cdot j} & T
\end{array}$$

Therefore, the following commutative diagram holds.

$$\begin{array}{ccccc}
t.e.K(G, \tau) & \xrightarrow{(f \cdot j)^\#} & t.e.K(H/\ker(f) \times S^\perp, (f \cdot j)^*\tau) & \xrightarrow{i_1^\#} & \\
M.d.G \downarrow & & M.d._{H/\ker(f) \times S^\perp} \downarrow & & \\
char(G, \tau) & \xrightarrow{char(f \cdot j)} & char(H/\ker(f) \times S^\perp, (f \cdot j)^*\tau) & \xrightarrow{char(i_1)} &
\end{array}$$

$$\begin{array}{ccc}
\begin{array}{c} \xrightarrow{i_1^\#} \\ \downarrow M.d.H/\ker(f) \end{array} & t.e.K(H/\ker(f), i_1^*(f \cdot j)^*\tau) & \begin{array}{c} \xrightarrow{q^\#} \\ \downarrow M.d.H \end{array} t.e.K(H, f^*\tau) \\
\begin{array}{c} \xrightarrow{char(i_1)} \\ \downarrow M.d.H/\ker(f) \end{array} & char(H/\ker(f), i_1^*(f \cdot j)^*\tau) & \begin{array}{c} \xrightarrow{char(q)} \\ \downarrow M.d.H \end{array} char(H, f^*\tau)
\end{array}$$

The first and the third commutes from Theorem 4.48. The second commutes from Theorem 4.46.

From $char(f) = char(q) \circ char(i_1) \circ char(f \cdot j)$, we obtain the conclusion.

5 The quasi functor RL

5.1 Positive energy representations of LT

Representation theory for loop groups is well known ([PS]). Especially we have an explicit description of irreducible positive energy representations of loop groups of tori.

Let us recall that LT has a canonical decomposition $LT \cong T \times \Pi_T \times U$, where T is the set of initial values of loops, Π_T is the set of “rotation numbers” (naturally isomorphic to $\pi_1(T) \cong \pi_0(LT)$) and

$$U := \exp \left\{ \beta : S^1 \rightarrow \mathfrak{t} \left| \int_{S^1} \beta(s) ds = 0 \right. \right\}$$

is the set of derivatives of contractible loops whose initial values are 0.

Let τ be a positive central extension of LT . The above decomposition is inherited partially, that is, $LT^\tau \cong (T \times \Pi_T)^\tau \otimes U^\tau$.

U^τ is called a Heisenberg group and has the unique irreducible positive energy representation $\rho_{\mathcal{H}} : U^\tau \rightarrow U(V_{\mathcal{H}}(U^\tau))$ at level τ up to equivalence. Concretely, $V_{\mathcal{H}}(U^\tau) := \widehat{S}((L\mathfrak{t}_{\mathbb{C}})_+)$ is a completion of the symmetric tensor algebra $S((L\mathfrak{t}_{\mathbb{C}})_+)$ of $(L\mathfrak{t}_{\mathbb{C}})_+$, which we call Heisenberg representation.

Remark 5.1. *The differential representation of it can be regarded as the “completion” of the Shrödinger representation of the infinite dimensional Heisenberg Lie algebra.*

When we choose a character $\lambda \in \Lambda_T^\tau$, $U^\tau \otimes T^\tau$ acts on $V_{\mathcal{H}}(U^\tau) \otimes \mathbb{C}_\lambda$ by

$$\rho_{\mathcal{H}} \otimes \lambda(u \otimes t)(v \otimes z) := \rho_{\mathcal{H}}(u)(v) \otimes \lambda(t)z,$$

where $u \in U^\tau$, $t \in T^\tau$, $v \in V_{\mathcal{H}}(U^\tau)$ and $z \in \mathbb{C}_\lambda$.

Then $LT^\tau = U^\tau \otimes (T \times \Pi_T)^\tau$ acts on

$$V_{[\lambda]} := \sum_{n \in \Pi_T} V_{\mathcal{H}}(U^\tau) \otimes \mathbb{C}_{\lambda + \kappa^\tau(n)}$$

where Π_T permutes the components. The isomorphism classes of irreducible positive energy representations are in 1:1 correspondence with the points of $\Lambda_T^\tau / \kappa^\tau(\Pi_T)$ ([PS] Proposition 9.5.11). Moreover,

$$l.w.T(V_{[\lambda]}) = [\lambda]_T.$$

5.2 Direct product

Let T_1 and T_2 be tori, τ_1 and τ_2 be positive central extensions of LT_1 and LT_2 respectively, i_1 be the natural inclusion into the first factor of $T_1 \times T_2$, and $p_j : T_1 \times T_2 \rightarrow T_j$ be the natural projection onto the j 'th factor ($j = 1, 2$). Then $T := T_1 \times T_2$ is a torus and $\tau := p_1^* \tau_1 + p_2^* \tau_2$ is a positive central extension of $LT \cong LT_1 \times LT_2$.

Using the above description, we will construct $i_1^! : R^\tau(LT) \rightarrow R^{i_1^* \tau}(LT_1)$ and verify that the following diagram commutes.

$$\begin{array}{ccc} RL(T, \tau) & \xrightarrow{i_1^!} & RL(T_1, i_1^* \tau) \\ l.w.T \downarrow & & l.w.T_1 \downarrow \\ char(T, \tau) & \xrightarrow{char(i_1)} & char(T_1, i_1^* \tau) \end{array}$$

In this situation, we have a natural isomorphisms $LT \cong LT_1 \times LT_2$ and $LT^\tau \cong LT_1^{\tau_1} \otimes LT_2^{\tau_2}$, that is, for any $l_1, l'_1 \in LT_1^{\tau_1}$ and $l_2, l'_2 \in LT_2^{\tau_2}$, $l_1 \otimes l_2 \cdot l'_1 \otimes l'_2 = l_1 l'_1 \otimes l_2 l'_2$. Moreover, $LT^\tau \cong U_1^{\tau_1} \otimes U_2^{\tau_2} \otimes (T_1 \times \Pi_1)^{\tau_1} \otimes (T_2 \times \Pi_2)^{\tau_2}$, where U_i is the group of contractible loops whose initial values are 0 of T_j ($j = 1, 2$).

From this commutativity, we can verify the following lemmas.

Lemma 5.2.

$$char(i_1)([\lambda]_T) = [{}^t di_1(\lambda)]_{T_1}.$$

Proof. Let us recall Lemma 1.13. That is, $\kappa^\tau = {}^t dp_1 \circ \kappa^{\tau_1} \circ dp_1 + {}^t dp_2 \circ \kappa^{\tau_2} \circ dp_2$ when $\tau = p_1^* \tau_1 + p_2^* \tau_2$.

From this formula, we have a bijection for any $\lambda \in \Lambda_T^\tau$

$$\begin{aligned} & \{ {}^t di_1(\lambda) + \kappa^{\tau_1}(n_1) | n_1 \in \Pi_{T_1} \} \times \{ {}^t di_2(\lambda) + \kappa^{\tau_2}(n_2) | n_2 \in \Pi_{T_2} \} \\ & \xrightarrow{{}^t dp_1 + {}^t dp_2} \{ {}^t dp_1 \circ {}^t di_1(\lambda) + {}^t dp_2 \circ {}^t di_2(\lambda) + {}^t dp_1 \circ \kappa^{\tau_1} \circ dp_1(n) + {}^t dp_2 \circ \kappa^{\tau_2} \circ dp_2(n) | n \in \Pi_T \} \\ & = \{ \lambda + \kappa^\tau(n) | n \in \Pi_T \}. \end{aligned}$$

Since ${}^t di_1 \circ {}^t dp_2 = 0$, we obtain the conclusion. \square

Lemma 5.3. Let $V_{\mathcal{H}}(U^\tau)$ be the Heisenberg representation space of U^τ . Then

$$V_{\mathcal{H}}(U^\tau) \cong V_{\mathcal{H}}(U_1^{\tau_1}) \otimes V_{\mathcal{H}}(U_2^{\tau_2})$$

as representation spaces.

Proof. This lemma follows from $U^\tau \cong U_1^{\tau_1} \otimes U_2^{\tau_2}$ and the product formula of the symmetric algebra

$$\begin{aligned} S((Lt_{\mathbb{C}})_+) & \cong S((L(\mathbf{t}_1 \oplus \mathbf{t}_2)_{\mathbb{C}})_+) \cong S((L\mathbf{t}_{1\mathbb{C}})_+ \oplus (L\mathbf{t}_{2\mathbb{C}})_+) \\ & \cong S((L\mathbf{t}_{1\mathbb{C}})_+) \otimes S((L\mathbf{t}_{2\mathbb{C}})_+). \end{aligned}$$

Factorization as representation spaces follows from the commutativity between $LT_1^{\tau_1}$ and $LT_2^{\tau_2}$ described above. \square

Lemma 5.4. *Let $\lambda \in \Lambda_T^\tau$ be a τ -twisted character of $T^\tau \cong T_1^{\tau_1} \otimes T_2^{\tau_2}$, then*

$$\mathbb{C}_\lambda \cong \mathbb{C}_{t_{di_1}(\lambda)} \otimes \mathbb{C}_{t_{di_2}(\lambda)}$$

as representation spaces.

Proof. $t = t_1 \otimes t_2 \in T^\tau$ is also written as $\tilde{i}_1(t_1) \cdot \tilde{i}_2(t_2)$, where $\tilde{i}_k : T_k^{\tau_k} \rightarrow T^\tau$ is induced from $i_k : T_k \rightarrow T$ ($k = 1, 2$). The following computation implies the statement.

$$\begin{aligned} \lambda(t) &= \lambda(\tilde{i}_1(t_1) \cdot \tilde{i}_2(t_2)) = \lambda(\tilde{i}_1(t_1))\lambda(\tilde{i}_2(t_2)) \\ &= (i_1^* \lambda)(t_1)(i_2^* \lambda)(t_2) \end{aligned}$$

When we regard the set of twisted character $\Lambda_{T_j}^{\tau_j}$ as a subset of $\text{Hom}(\Pi_{T_j^{\tau_j}}, \mathbb{Z})$, $i_1^* \lambda$ corresponds to $t_{di_1}(\lambda)$. \square

The above lemmas imply the following theorem.

Theorem 5.5. *We have an isomorphism*

$$V_{[\lambda]} \cong V_{[t_{di_1}(\lambda)]} \otimes V_{[t_{di_2}(\lambda)]}$$

as representation spaces.

Proof.

$$\begin{aligned} V_{[\lambda]} &= \sum_{n \in \Pi_T} V_{\mathcal{H}}(U^\tau) \otimes \mathbb{C}_{\lambda + \kappa^\tau(n)} \\ &\cong \sum_{n_1 \in \Pi_{T_1}, n_2 \in \Pi_{T_2}} V_{\mathcal{H}}(U_1^{\tau_1}) \otimes V_{\mathcal{H}}(U_2^{\tau_2}) \otimes \mathbb{C}_{t_{di_1}(\lambda) + \kappa^{\tau_1}(n_1)} \otimes \mathbb{C}_{t_{di_2}(\lambda) + \kappa^{\tau_2}(n_2)} \\ &\cong \sum_{n_1 \in \Pi_{T_1}, n_2 \in \Pi_{T_2}} [V_{\mathcal{H}}(U_1^{\tau_1}) \otimes \mathbb{C}_{t_{di_1}(\lambda) + \kappa^{\tau_1}(n_1)}] \otimes [V_{\mathcal{H}}(U_2^{\tau_2}) \otimes \mathbb{C}_{t_{di_2}(\lambda) + \kappa^{\tau_2}(n_2)}] \\ &\cong \left[\sum_{n_1 \in \Pi_{T_1}} V_{\mathcal{H}}(U_1^{\tau_1}) \otimes \mathbb{C}_{t_{di_1}(\lambda) + \kappa^{\tau_1}(n_1)} \right] \otimes \left[\sum_{n_2 \in \Pi_{T_2}} V_{\mathcal{H}}(U_2^{\tau_2}) \otimes \mathbb{C}_{t_{di_2}(\lambda) + \kappa^{\tau_2}(n_2)} \right] \\ &= V_{[t_{di_1}(\lambda)]} \otimes V_{[t_{di_2}(\lambda)]} \end{aligned}$$

Factorization as representation spaces follows from Lemma 5.3, 5.4 and the bijection in the proof of Lemma 5.2. \square

Since the dimension of $V_{[t_{di_2}(\lambda)]}$ is infinite, the following follows immediately.

Corollary 5.6. *When we regard an irreducible representation of LT^τ as a representation of $LT_1^{\tau_1}$, it is never finitely reducible if $\dim T_2 \geq 1$.*

Motivated by this observation, we define the new “induced” representation $i_1^! V_{[\lambda]}$ so that it has the compatibility with $\text{char}(i_1)$.

Definition 5.7. Let V be a finitely reducible representation space of LT^τ .

$$\begin{aligned} i_1^! V &:= \sum_{[\lambda_2] \in \Lambda_{T_2}^{\tau_2} / \kappa^{\tau_2}(\Pi_{T_2})} \text{Hom}_{LT_2^{\tau_2}}(V_{[\lambda_2]}, i_2^* V) \\ &\cong \text{Hom}_{LT_2^{\tau_2}}\left(\sum_{[\lambda_2] \in \Lambda_{T_2}^{\tau_2} / \kappa^{\tau_2}(\Pi_{T_2})} V_{[\lambda_2]}, i_2^* V\right) \end{aligned}$$

where, $\text{Hom}_{LT_2^{\tau_2}}(V_{[\lambda_2]}, i_2^* V)$ is the set of bounded intertwining operators. The topology of the left hand side is defined below.

Proposition 5.8. $i_1^! V$ is a representation space of $LT_1^{\tau_1}$. The action is defined by

$$(l.F)(v) := l.(F(v)),$$

where $F \in i_1^! V$, $l \in LT_1^{\tau_1}$ and $v \in \sum_{[\lambda_2] \in \Lambda_{T_2}^{\tau_2} / \kappa^{\tau_2}(\Pi_{T_2})} V_{[\lambda_2]}$.

Proof. It is sufficient to verify that $l.F$ is an intertwining operator. From the linearity of $l.F$, we can assume that $v \in V_{[\lambda_2]}$ for some $[\lambda_2] \in \Lambda_{T_2}^{\tau_2} / \kappa^{\tau_2}(\Pi_{T_2})$. Let $l \in LT_1^{\tau_1}$, $l' \in LT_2^{\tau_2}$ and $F \in \text{Hom}_{LT_2^{\tau_2}}(V_{[\lambda_2]}, i_2^* V)$.

$$\begin{aligned} (l.F)(l'.v) &= l.(F(l'.v)) = l.(l'.(F(v))) \\ &= (ll').F(v) = (l'l).F(v) \\ &= l'.(l.(F(v))) = l'.(l.F)(v). \end{aligned}$$

The first and the last equality follow from the definition. The second one follows from that F is an intertwining operator. The third and the fifth one follow from that a representation is a group homomorphism. The fourth one follows from the commutativity described above. \square

The following lemma is clear from the definition.

Lemma 5.9. For any finitely reducible positive energy representation space V_1 and V_2 of LT^τ at level τ ,

$$i_1^!(V_1 \oplus V_2) \cong i_1^! V_1 \oplus i_1^! V_2.$$

From this lemma, we can assume that V is irreducible.

Theorem 5.10.

$$i_1^! V_{[\lambda]} \cong V_{[t_{di_1}(\lambda)]},$$

therefore, $i_1^!$ has a compatibility with $\text{char}(i_1)$.

If we define an inner product of the left hand side can be defined by $(F_1, F_2) := (F_1(v), F_2(v))_{V_{[\lambda]}}$, where v is a chosen unit vector in $V_{[t_{di_2}(\lambda)]}$, this inner product does not depend on the choice of v , and the above isomorphism is isometric.

Proof. Fix a completely orthonormal system $\{w_j\}_{j \in \mathbb{N}}$ of $V_{[{}^t di_1(\lambda)]}$. From Theorem 5.5, $V_{[\lambda]} \cong \sum_j \mathbb{C}w_j \otimes V_{[{}^t di_2(\lambda)]}$ as a representation space of $LT_2^{\tau_2}$. So we can decompose $F \in \text{Hom}_{LT_2^{\tau_2}}(V_{[\lambda_2]}, i_2^* V_{[\lambda]})$ as

$$F = \sum_j F_j,$$

where $F_j : V_{[{}^t di_2(\lambda)]} \rightarrow \mathbb{C}w_j \otimes V_{[{}^t di_2(\lambda)]} \cong V_{[{}^t di_2(\lambda)]}$.

By Schur's lemma, $F_j = c_j \text{id}$ for some $c_j \in \mathbb{C}$. Since F determines an operator, for any $v \in V_{[{}^t di_2(\lambda)]}$, $\sum_j F_j(v) = v \otimes \sum_j c_j w_j \in V_{[\lambda]}$. Therefore, $\sum_j |c_j|^2 < \infty$.

Moreover, by Schur's lemma, if $[\lambda_2] \neq [{}^t di_2(\lambda)]$, $\text{Hom}_{LT_2^{\tau_2}}(V_{[\lambda_2]}, i_2^* V_{[\lambda]}) = 0$. Therefore, we can define an isomorphism

$$\sum_{[\lambda] \in \Lambda_{T_2}^{\tau_2} / \kappa^{\tau_2}(\Pi_{T_2})} \text{Hom}_{LT_2^{\tau_2}}(V_{[\lambda_2]}, i_2^* V_{[\lambda]}) = \text{Hom}_{LT_2^{\tau_2}}(V_{[{}^t di_2(\lambda)]}, i_2^* V_{[\lambda]}) \rightarrow V_{[i_1^* \lambda]}$$

by

$$F \mapsto \sum_j c_j w_j.$$

Since $\sum_j |c_j|^2 < \infty$, the infinite sum $\sum_j c_j w_j$ converges.

Moreover, since $\sum_j c_j w_j$ is determined by $F(v) = v \otimes \sum_j c_j w_j$, this isomorphism is independent of the choice of v and completely orthonormal system. \square

From this theorem, we obtain the following conclusion.

Corollary 5.11. *The following commutative diagram holds.*

$$\begin{array}{ccc} RL(T, \tau) & \xrightarrow{i_1^\dagger} & RL(T', i_1^* \tau) \\ l.w.T \downarrow & & l.w.T_1 \downarrow \\ char(T, \tau) & \xrightarrow{char(i_1)} & char(T_1, i_1^* \tau) \end{array}$$

5.3 Finite covering

In this section, we study in the case of a finite covering.

Let $q : T' \rightarrow T$ be a finite covering. We can compute $char(q)$ from Lemma 1.12 with the same way in Lemma 5.2.

Lemma 5.12.

$$char(q)([\lambda]_T) = \sum_{[m] \in \Pi_T / dq(\Pi_{T'})} [{}^t dq(\lambda + \kappa^\tau(m))]_{T'},$$

where m is a chosen representative element of $[m]$.

Remark 5.13. $\Pi_{T'}$ -orbit $[{}^t dq(\lambda) + \kappa^\tau(dq(m))]\pi_{T'}$ is independent of a choice of representative element m .

Proof. It follows from the following.

$$\begin{aligned} & \{{}^t dq(\lambda + \kappa^\tau(n)) | n \in \Pi_T\} = \{{}^t dq(\lambda) + {}^t dq(\kappa^\tau(n)) | n \in \Pi_T\} \\ &= \{{}^t dq(\lambda) + {}^t dq(\kappa^\tau(dq(n') + m)) | n' \in \Pi_{T'}, m \in \Pi_T/dq(\Pi_{T'})\} \\ &= \coprod_{m \in \Pi_T/dq(\Pi_{T'})} \{{}^t dq(\lambda + \kappa^\tau(m)) + \kappa^{q^*\tau}(n') | n' \in \Pi_{T'}\}. \end{aligned}$$

□

Let us define the induced homomorphism $q^!$.

Definition 5.14. Let V be a finitely reducible representation space of LT^τ .

$$q^!V := q^*V,$$

where the action of $LT'^{q^*\tau}$ on q^*V is defined through the homomorphism Lq .

The following lemma is clear from the definition.

Lemma 5.15. For any finitely reducible positive energy representation spaces V_1 and V_2 of LT^τ at level τ ,

$$q^!(V_1 \oplus V_2) \cong q^!V_1 \oplus q^!V_2.$$

From this lemma, we can assume that V is irreducible.

Theorem 5.16.

$$q^!(V_{[\lambda]}) \cong \sum_{m \in \Pi_T/dq(\Pi_{T'})} V_{[{}^t dq(\lambda + \kappa^\tau(m))]}.$$

Moreover, the following commutative diagram holds.

$$\begin{array}{ccc} R^\tau(LT) & \xrightarrow{q^!} & R^{q^*\tau}(LT') \\ \downarrow l.w.T & & \downarrow l.w.T' \\ char(T, \tau) & \xrightarrow{char(q)} & char(T', q^*\tau) \end{array}$$

Proof. Since T and T' are locally isomorphic, the set of the derivatives of contractible loops whose initial values are 0 of T corresponds to one of T' by the natural way. That is, $LT^\tau \cong (T \times \Pi_T)^\tau \otimes U^\tau$, $LT'^{q^*\tau} \cong (T' \times \Pi_{T'})^{q^*\tau} \otimes U^{q^*\tau}$ and $U^\tau \cong U^{q^*\tau}$. Therefore, the Heisenberg representations $V_{\mathcal{H}}$ of $U^\tau \subseteq LT^\tau$ coincide with one of $U^{q^*\tau} \subseteq LT'^{q^*\tau}$.

$$q^*(V_{[\lambda]}) = q^*\left(\sum_{n \in \Pi_T} V_{\mathcal{H}} \otimes \mathbb{C}_{\lambda + \kappa^\tau(n)}\right) = \sum_{n \in \Pi_T} V_{\mathcal{H}} \otimes \mathbb{C}_{{}^t dq(\lambda + \kappa^\tau(n))}$$

$$\begin{aligned}
&= \sum_{m \in \Pi_T / dq(\Pi_{T'})} \sum_{n' \in \Pi_{T'}} V_{\mathcal{H}} \otimes \mathbb{C}_{\iota dq(\lambda + \kappa^\tau(m)) + \kappa^{q^* \tau}(n')} \\
&= \sum_{m \in \Pi_T / dq(\Pi_{T'})} V_{[\iota dq(\lambda + \kappa^\tau(m))]}
\end{aligned}$$

Commutativity of the above diagram is clear from this description and Lemma 3.4. \square

5.4 Local injection

Let $f : T' \rightarrow T$ be a local injection. From Theorem 3.10, we can decompose f as the following sequence

$$T' \xrightarrow{q} T' / \ker(f) \xrightarrow{i_1} T' / \ker(f) \times T^\perp \xrightarrow{f \cdot j} T,$$

where q and $f \cdot j$ are finite coverings, i_1 is the natural inclusion into the first factor.

Definition 5.17.

$$f^! := q^! \circ i_1^! \circ (f \cdot j)^!$$

The following theorem follows from Corollary 5.11, Theorem 5.16, Theorem 3.10 and the same argument in the proof of Theorem 4.43.

Theorem 5.18. *The following commutative diagram holds.*

$$\begin{array}{ccc}
R^\tau(LT) & \xrightarrow{f^!} & R^{f^* \tau}(LT') \\
l.w.T \downarrow & & l.w.T' \downarrow \\
char(T, \tau) & \xrightarrow{char(f)} & char(T', f^* \tau)
\end{array}$$

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